Dynamic systems and their behaviour

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Literature

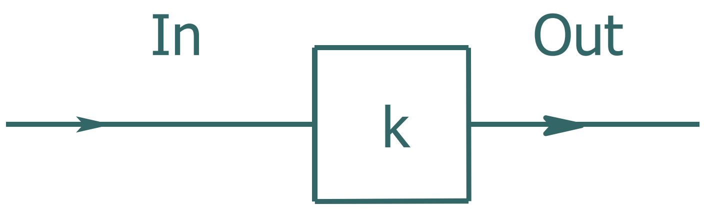
- J. Sanny and W. Moebs, University Physics, Wm. C. Brown Publishers

- L. Meirovitch, Fundamentals of vibrations, Mc.Graw-Hill.
- S.S. Rao, Mechanical Vibrations, Addison-Wesley
- Wikipedia: http://en.wikipedia.org/wiki/Vibration#Free_vibration_without_damping

Zero order systems

$$a_0 \cdot Out(t) = b_0 \cdot In(t)$$

$$\frac{Out(t)}{In(t)} = \frac{b_0}{a_0} = k$$



- Examples:
  - Resistor (ideal): V(t)= R*I(t)
  - Spring (ideal): F(t)= k*x(t)

First order systems

$$a_1 \cdot \frac{dOut(t)}{dt} + a_0 \cdot Out(t) = b_0 \cdot In(t)$$

$$\frac{a_1}{a_0} \cdot \frac{dOut(t)}{dt} + Out(t) = \frac{b_0}{a_0} \cdot In(t)$$

$$\frac{a_1}{a_0} = \text{time constant}\,[\text{sec}]$$

- Examples (energy storing):
  - Temperature sensor put in a heated bath
  - Room heating
  - (Dis)charging battery

Temperature sensor put in a cooled bath

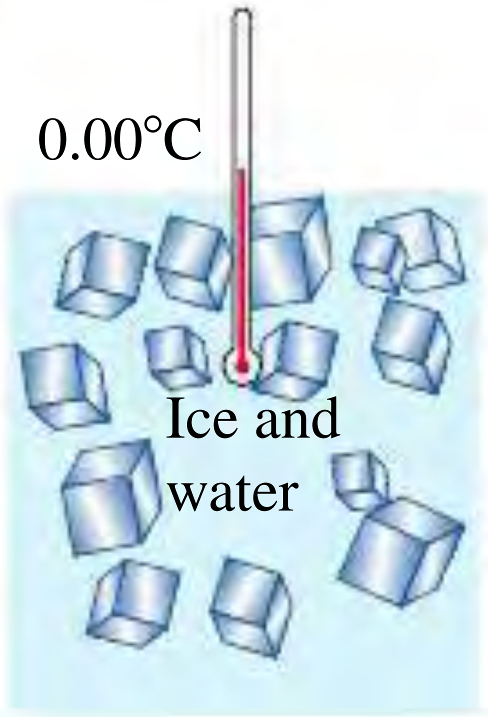
$$Q_{Add} - Q_{Loss} = Q_{Heat-up}$$

$$Q_{Loss} = 0$$

$$Q_{Add} = \alpha A\left(T_e - T_t\right)$$

$$Q_{Heat-up} = mC\frac{dT_t}{dt}$$

$$\alpha A\left(T_e - T_t\right) = mC\frac{dT_t}{dt} = -mC\frac{d\left(T_e - T_t\right)}{dt}$$



Solution (for the step response)

$$T_e - T_t = \left(T_e - T_{t_O}\right) * e^{\frac{-t}{\tau}}$$

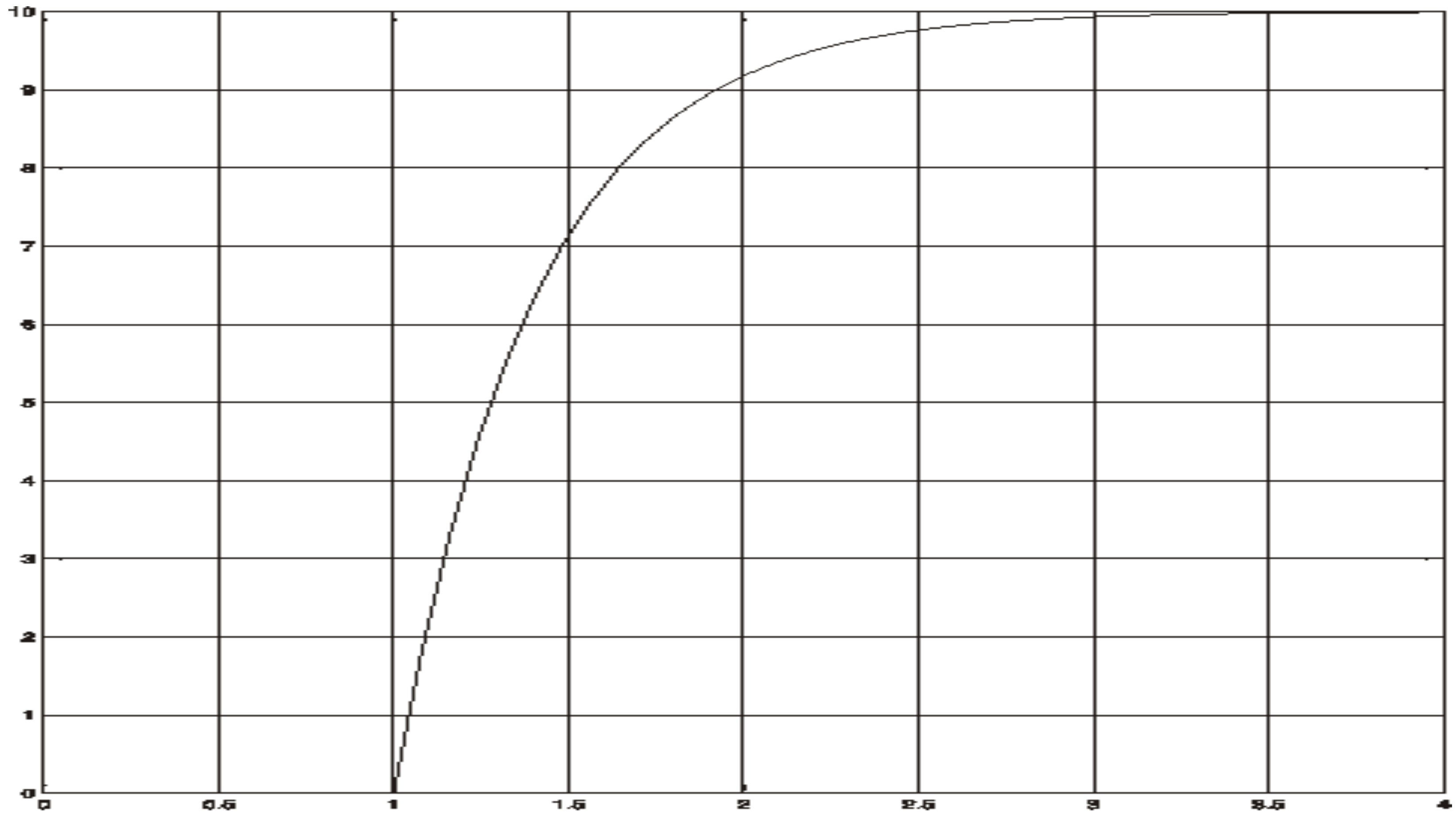
$$T_t = T_e - \left(T_e - T_{t_O}\right) * e^{\frac{-t}{\tau}}$$

$$\tau = \frac{m * c}{\alpha * A}$$

$$T_{t_O} = \text{sensor temperature at } t = 0$$

Example: room temperature in Simulink

Step response first order system



Response to sine input

$$\tau \frac{dOut(t)}{dt} + Out(t) = k \cdot In(t)$$

$$In(t) = I_0 e^{st}$$

$$\text{Assume}: Out(t) = O_0 e^{st}$$

$$\tau s O_0 e^{st} + O_0 e^{st} = O_0 e^{st}(\tau s + 1) = k \cdot I_0 e^{st}$$

$$\frac{Out(t)}{In(t)} = \frac{k}{\tau s + 1}$$

$$s = i \cdot \omega$$

$$\frac{Out(t)}{In(t)} = \frac{k}{\tau \cdot i \cdot \omega + 1}$$

Second order systems

$$a_2 \cdot \frac{d^2Out(t)}{dt} + a_1 \cdot \frac{dOut(t)}{dt} + a_0 \cdot Out(t) = b_0 \cdot In(t)$$

$$\frac{a_2}{a_0} \cdot \frac{d^2Out(t)}{dt} + \frac{a_1}{a_0} \cdot \frac{dOut(t)}{dt} + Out(t) = \frac{b_0}{a_0} \cdot In(t)$$

$$\sqrt{\frac{a_2}{a_0}} = \omega_0 \text{ eigenfrequency of the system}$$

$$\gamma = \frac{a_1}{2\sqrt{a_0 \cdot a_2}} \text{ dampings factor}$$

Tacoma Narrows bridge

Free vibrations, unforced

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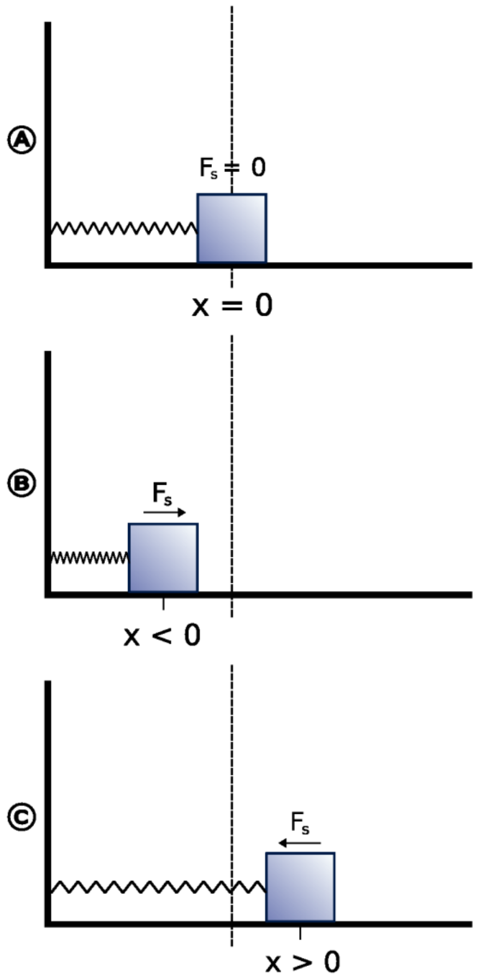
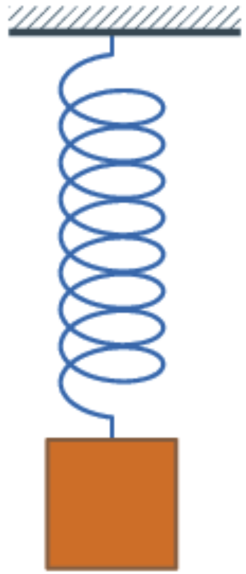
Simple Harmonic Motion

$$F(x) = -k \cdot x$$





Picture from Wikipedia

Equations of Simple Harmonic Motion

Newton's law, force balance:

$$-k \cdot x = m \cdot a$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Solution: $x(t) = A \cdot \sin\left(\sqrt{\frac{k}{m}} \cdot t + \phi\right)$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{dx}{dt} = A \cdot \sqrt{\frac{k}{m}} \cdot \cos\left(\sqrt{\frac{k}{m}} \cdot t + \phi\right)$$

$$\frac{d^2x}{dt^2} = -A \cdot \frac{k}{m} \cdot \sin\left(\sqrt{\frac{k}{m}} \cdot t + \phi\right)$$

$$-A \cdot \frac{k}{m} \cdot \sin\left(\sqrt{\frac{k}{m}} \cdot t + \phi\right) + \frac{k}{m} \cdot A \cdot \sin\left(\sqrt{\frac{k}{m}} \cdot t + \phi\right) = 0$$

Parameters of Simple Harmonic Motion

$$x(t) = A \cdot \sin(\omega \cdot t + \phi)$$

Angular frequency: $\omega = \sqrt{\frac{k}{m}}$ in rad/s

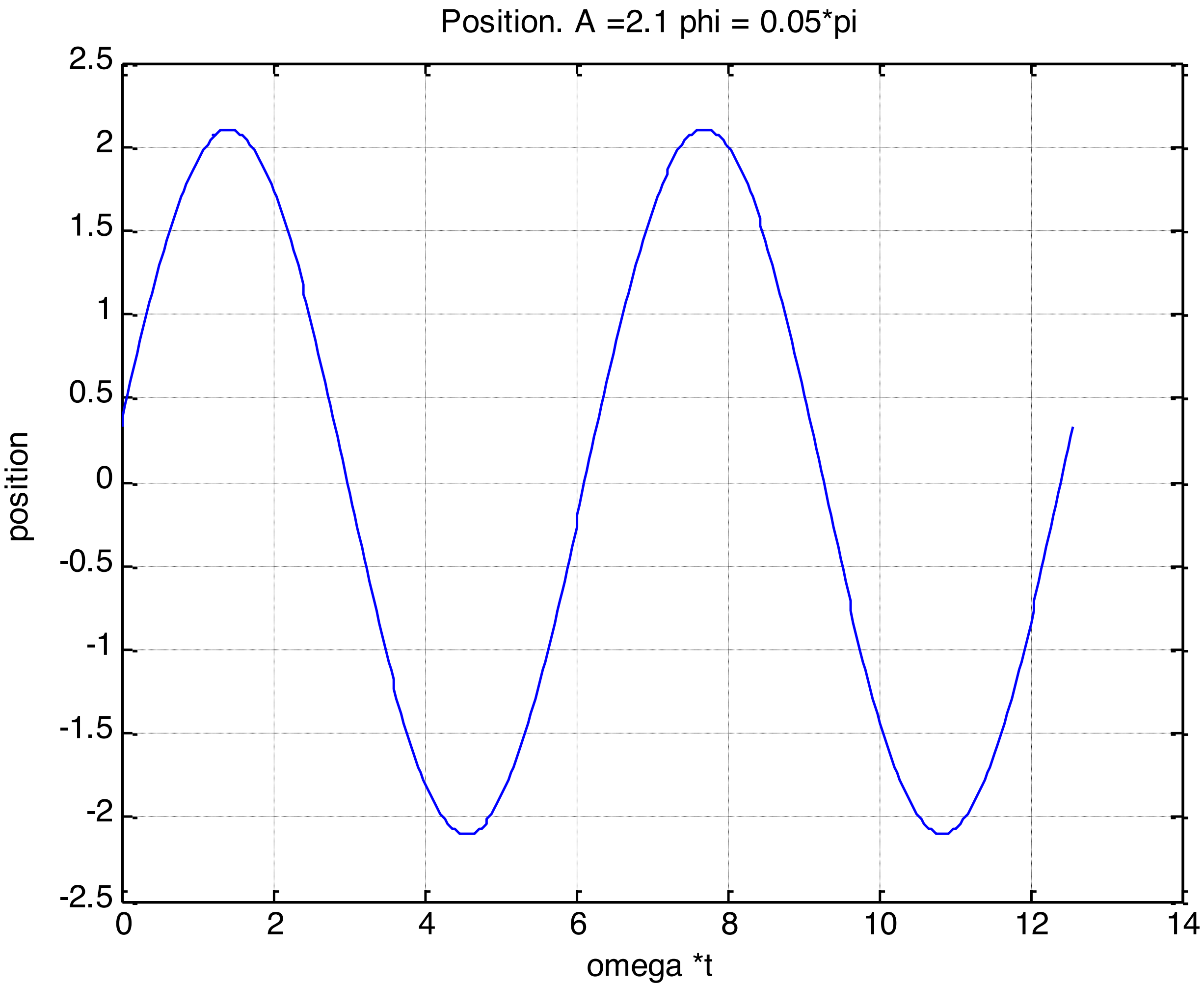
Frequency: $f = \frac{\omega}{2\pi}$

Phase: $\omega \cdot t + \phi$ in rad

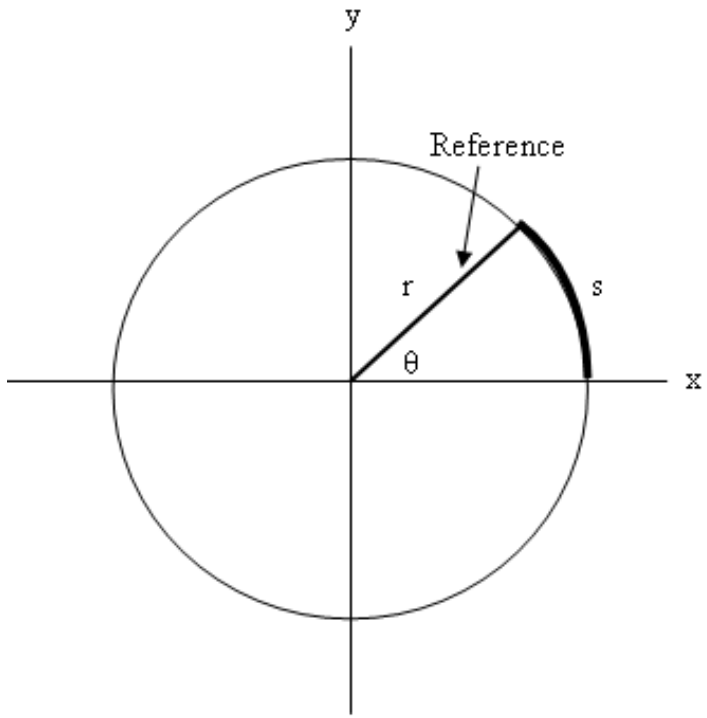
Amplitude: A

Period time: $T = \frac{2\pi}{\omega} = \frac{1}{f}$ Time required for one complete oscillation.

Solution graphically



Circular Motion and Simple Harmonic Motion



$$x(t) = r \cdot \cos(\theta(t))$$

$$y(t) = r \cdot \sin(\theta(t))$$

$$Stel : \theta(t) = \omega \cdot t + \phi$$

$$x(t) = r \cdot \cos(\omega \cdot t + \phi)$$

$$y(t) = r \cdot \sin(\omega \cdot t + \phi)$$

Grandfather clock

- The oscillations of the pendulum was used to keep time

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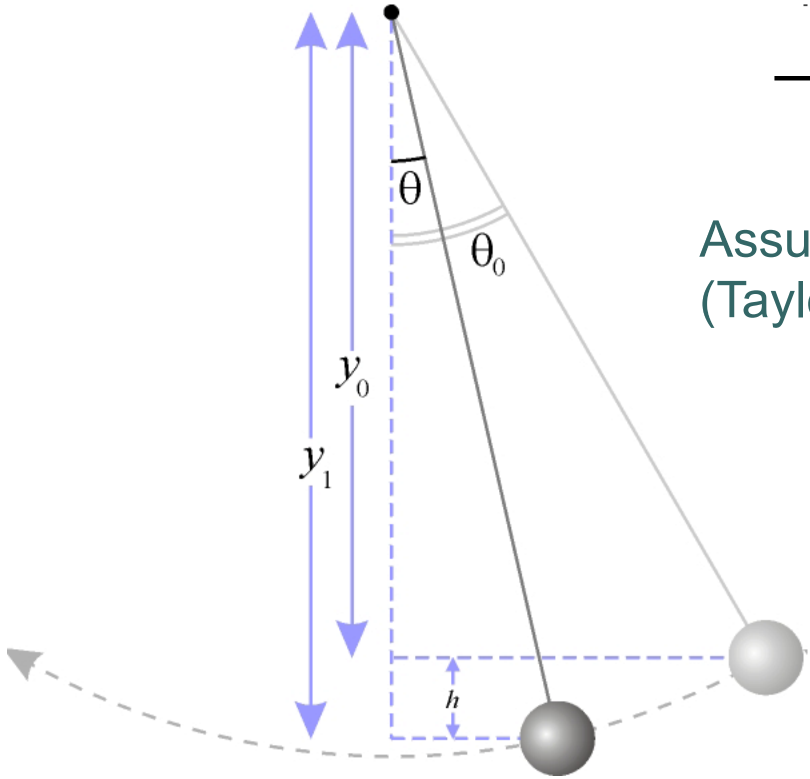
A Simple Pendulum,

Sum of all torques around rotation center:



$$-mgl\sin\theta = ml^2\frac{d^2\theta}{dt^2}$$

Assume small amplitudes of θ
(Taylor expansion) sin θ = θ

$$-mgl\theta = ml^2\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

Solution:

$$\theta(t) = A \cdot \sin(\omega \cdot t + \phi)$$

$$A = \text{amplitude}$$

$$\phi = \text{phase constant}$$

$$\omega = \text{angular frequency} = \sqrt{\frac{g}{l}}$$

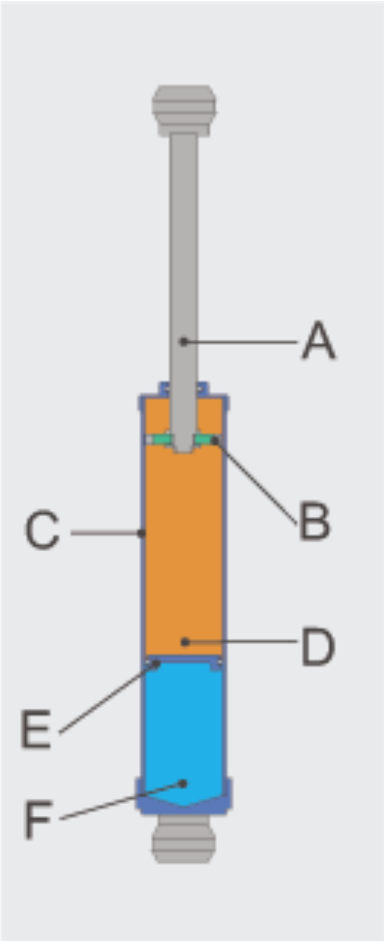
Compare with: $x(t) = A \cdot \sin(\omega \cdot t + \phi) \vee \omega = \sqrt{\frac{k}{m}}$

for translatory vibrations

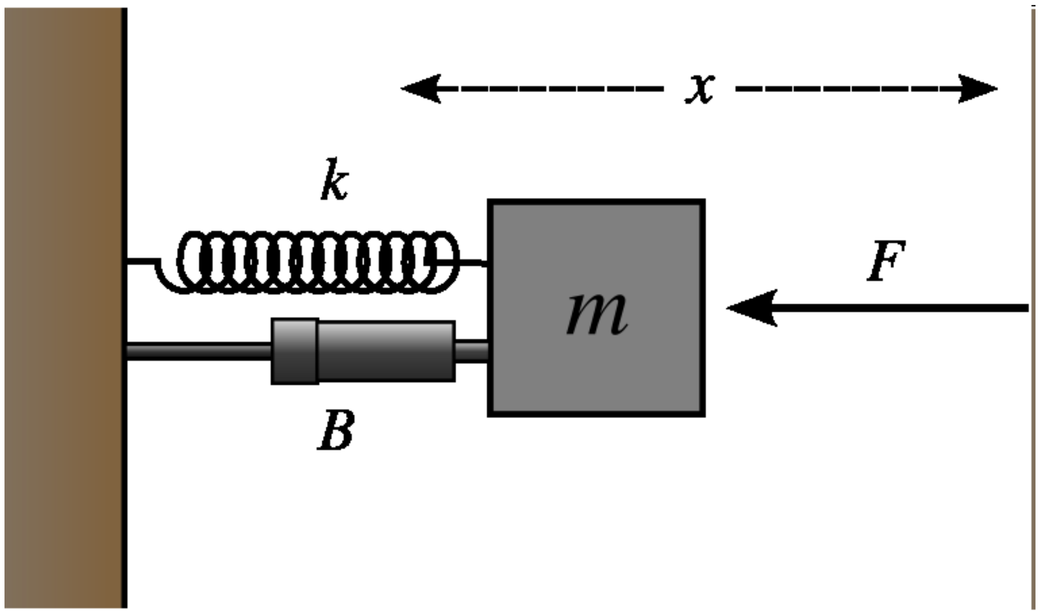
Damped Oscillations

- Real systems have damping for instance through friction.
- Since friction is a dissipative force the amplitude of oscillations must decrease with time.
- The frictional force is often caused by the medium in which the oscillating body is immersed.

Example: Gas damper



Mass-spring-damper system



Damped oscillations

$$F_{R=} = -b \cdot v = -b \cdot \frac{dx}{dt}$$

$$b = \text{damping constant}$$

$$\text{Newton's second law:}$$

$$\sum F_x = m \cdot a_x$$

$$-k \cdot x - b \cdot \frac{dx}{dt} = m \cdot \frac{d^2x}{dt^2}$$

$$m \cdot \frac{d^2x}{dt^2} + b \cdot \frac{dx}{dt} + k \cdot x = 0$$

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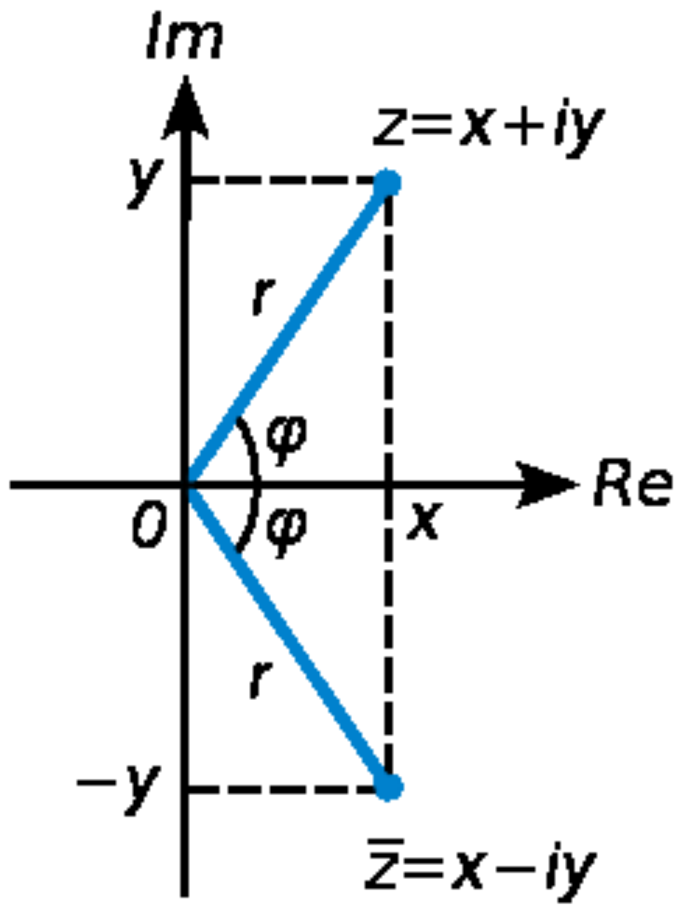
Complex numbers



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Solution using complex variables

Suppose the solution is of the form:

$$x(t) = A \cdot e^{(s \cdot t)}$$

$$v(t) = s \cdot A \cdot e^{(s \cdot t)}$$

$$a(t) = s^2 \cdot A \cdot e^{(s \cdot t)}$$

$$m \cdot \frac{d^2 x}{dt^2} + b \cdot \frac{dx}{dt} + k \cdot x = 0$$

$$\left(s^2 \cdot m + s \cdot b + k\right) \cdot A \cdot e^{(s \cdot t)} = 0$$

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$$s^2 + s \cdot \frac{b}{m} + \frac{k}{m} = 0$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\gamma = \frac{b}{2m}$$

$$s^2 + 2\cdot\gamma\cdot s + \omega_0^2 = 0$$

Quadratieq.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2\cdot\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} =$$

$$s_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

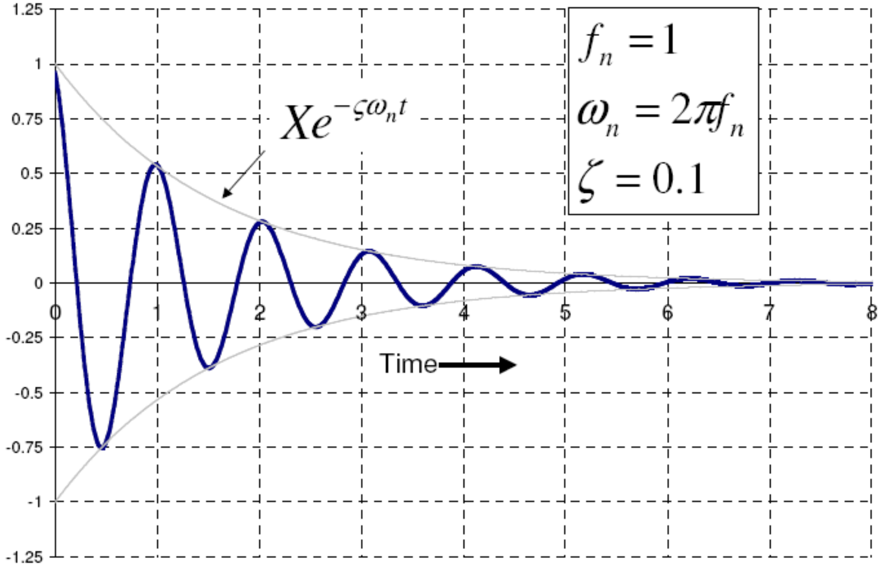
3 cases

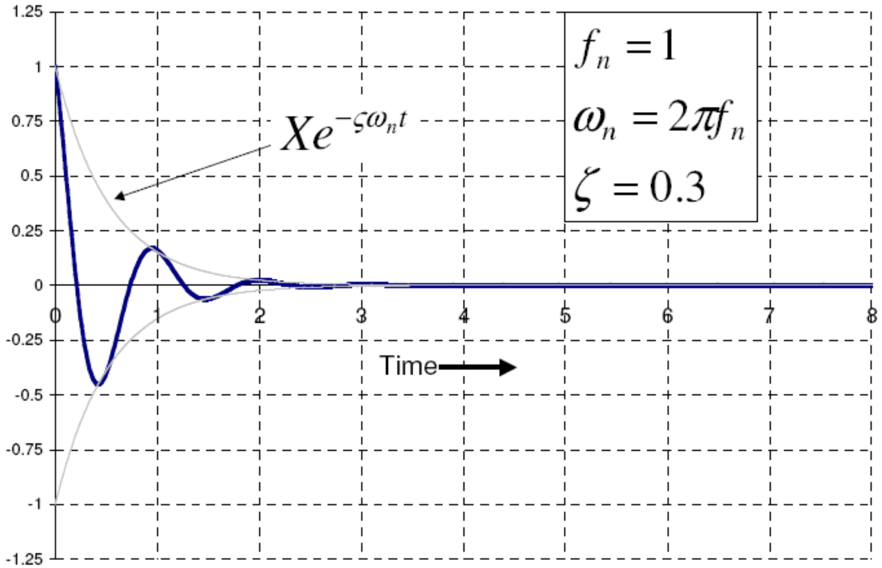
$$\text{Underdamped}: \ \gamma < 1$$

$$\omega_0 = \sqrt{\frac{k}{m}} > \gamma = \frac{b}{2m}$$

$$x(t) = A \cdot e^{-\gamma t} \cos(\omega' t + \phi)$$

$$\omega' = \sqrt{\omega_0^2 - \gamma^2}$$



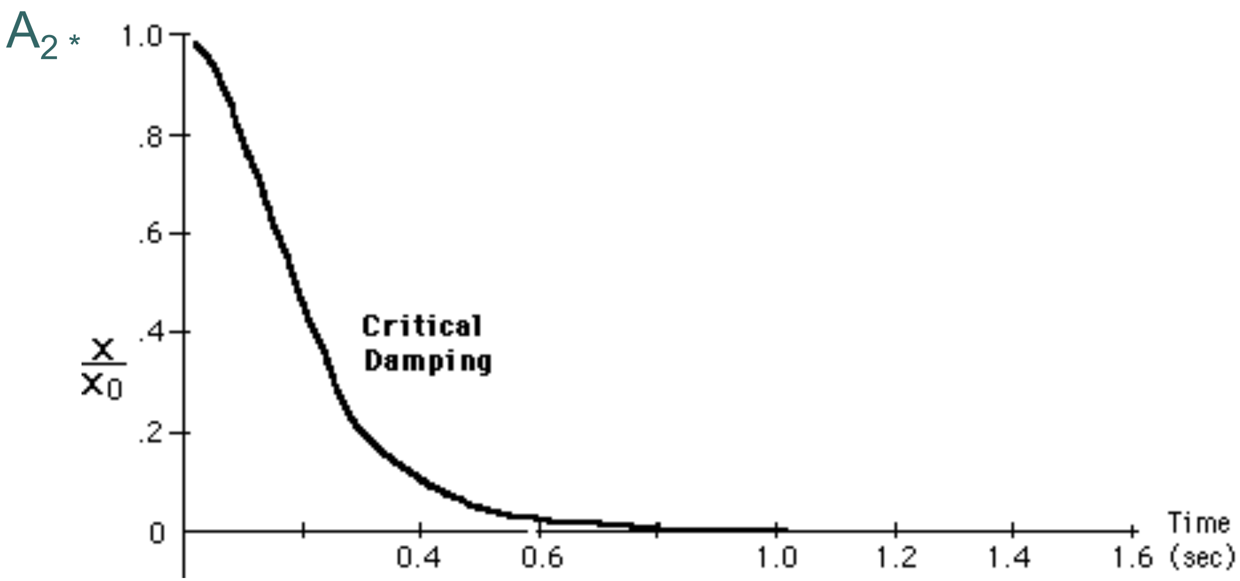


$$\zeta = \frac{b}{2\sqrt{k \cdot m}}$$

Critically damped

$$\gamma = 1$$

$$x(t) = e^{-\gamma \cdot t}\left(A_1 \cdot t + A_2\right)$$

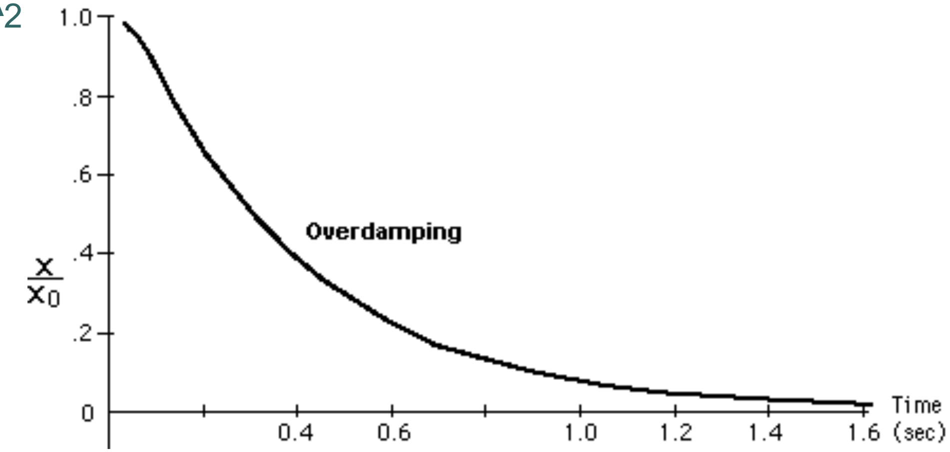


Overdamped

$$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}$$

$$-\gamma_{1,2} = \frac{b \pm (b^2 - 4km)^{1/2}}{2m}$$

$A_1 + A_2$



Forced Oscillations and Resonance

Fourier series

- Each periodic function (piecewise smooth, continuous and periodic) can be rewritten as:

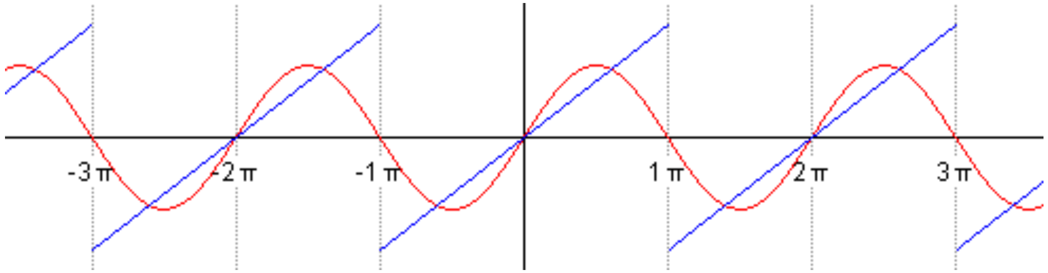
$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cdot \cos(\omega_n \cdot t) + b_n \cdot \sin(\omega_n \cdot t)\right]$$

$$\omega_n = n \cdot \frac{2\pi}{T}$$

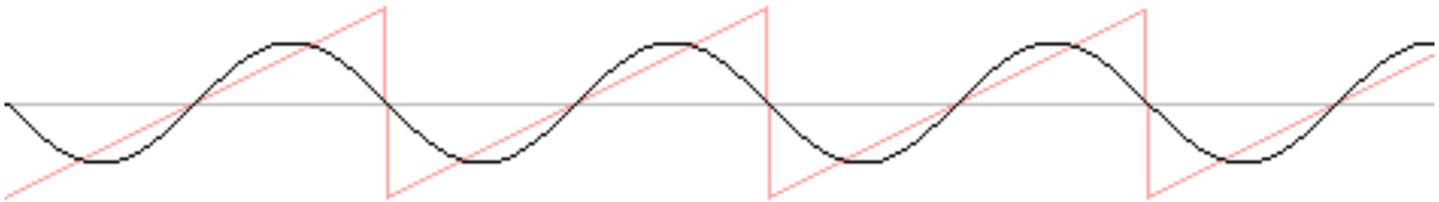
$$a_n = \frac{2}{T}\int_{t_1}^{t_2} f(t) \cdot \cos(\omega_n \cdot t)dt$$

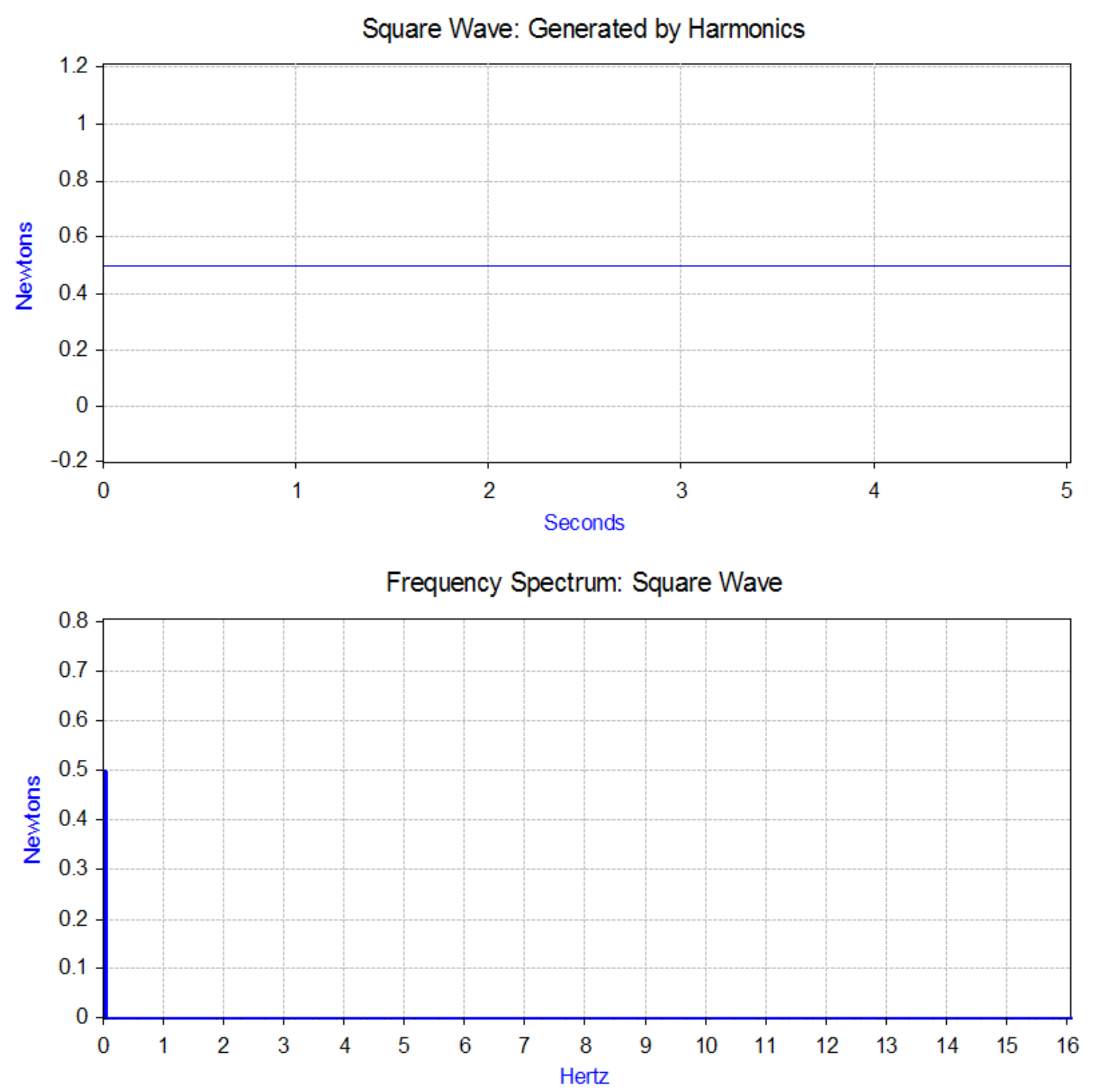
$$b_n = \frac{2}{T}\int_{t_1}^{t_2} f(t) \cdot \sin(\omega_n \cdot t)dt$$

Example sawtooth



harmonics: 1





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Thus

- When one knows how a system reacts to sine and cosine functions one knows how a system reacts to any periodic function!

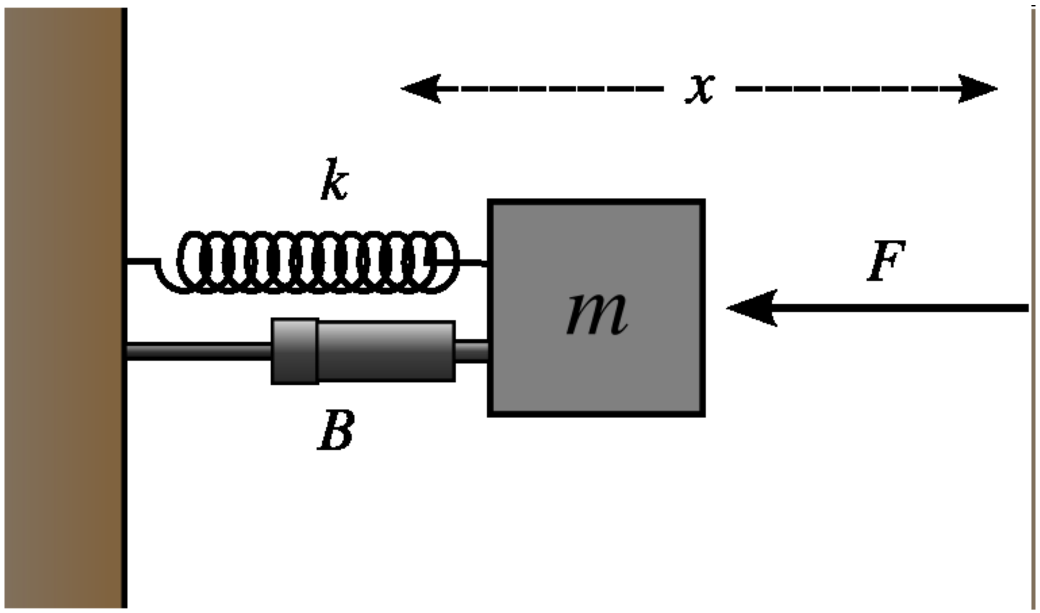
- Remark: A cosine function is a sine function with a phase difference of $\pi/2$

Fourier Transforms

- Mathematical software packages implement Fourier Transforms
  - Discrete Fourier Transform (DFT)
  - Fast Fourier Transform (FFT)

  - Matlab:
    - Help search Fourier
  - Mathematica
    - http://demonstrations.wolfram.com/ExamplesOfFourierSeries/

Mass-spring-damper system



Equation forced vibration

$$\sum F_x = m \cdot a_x$$

$$F(t) = F_{excitation} \cdot \cos(\omega_{excitation} \cdot t)$$

$$-k \cdot x - b \cdot \frac{dx}{dt} + F_{excitation} \cdot \cos(\omega_{excitation} \cdot t) = m \cdot \frac{d^2x}{dt^2}$$

$$m \cdot \frac{d^2x}{dt^2} + b \cdot \frac{dx}{dt} + k \cdot x = F_{excitation} \cdot \cos(\omega_{excitation} \cdot t)$$

Solution

$$x(t) = X \cdot \cos(\omega_{excitation} \cdot t - \phi)$$

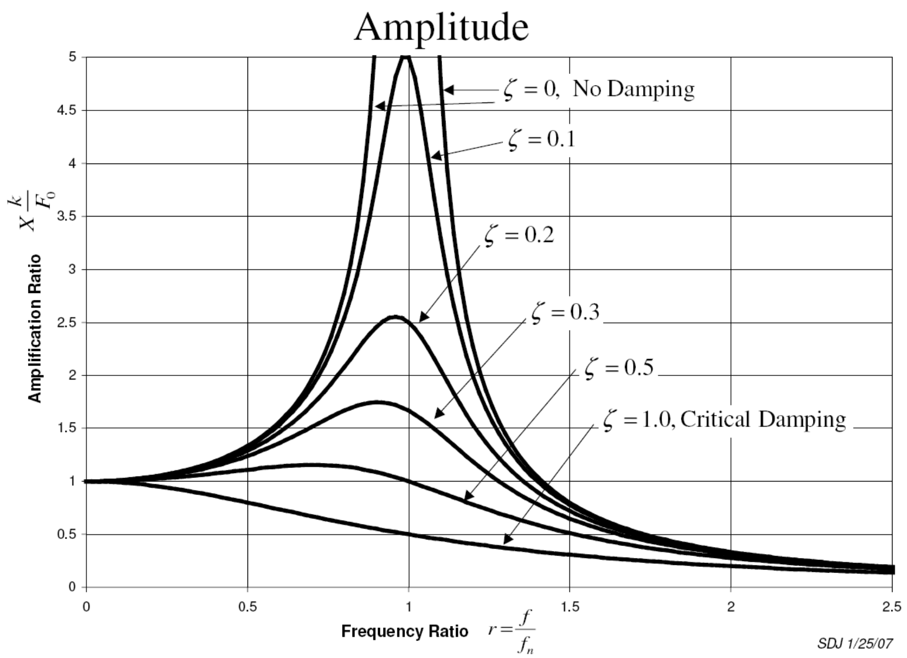
$$\zeta = \frac{b}{2\sqrt{k \cdot m}}$$

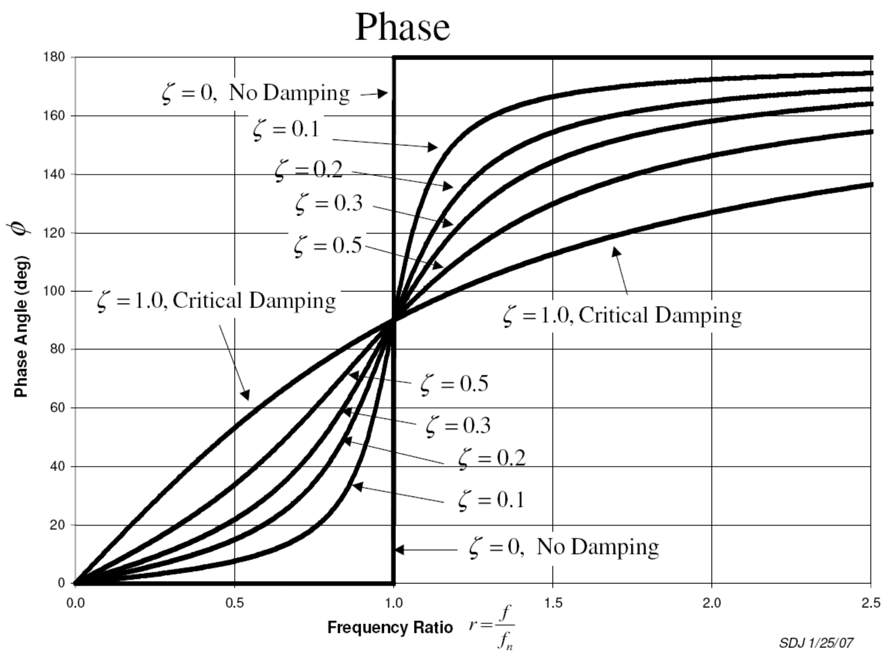
$$X = \frac{F_{excitation}}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2 \cdot \zeta \cdot r)^2}}$$

$$r = \frac{\omega_{excitation}}{\omega_{eigen}} = \frac{f_{excitation}}{f_{eigen}}$$

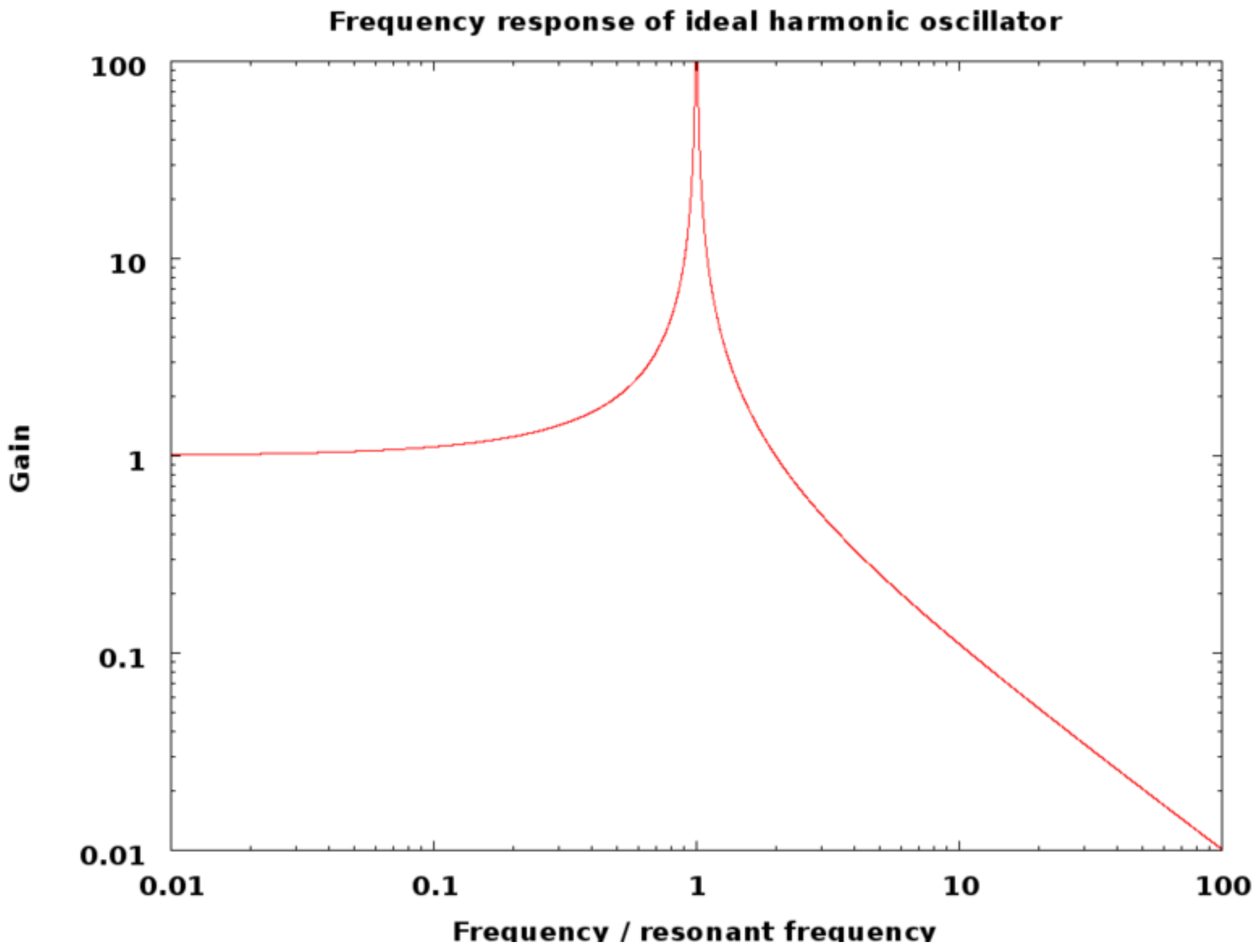
$$\phi = \arctan\left(\frac{2 \cdot \zeta \cdot r}{1-r^2}\right)$$

Forced response mass-spring damper system



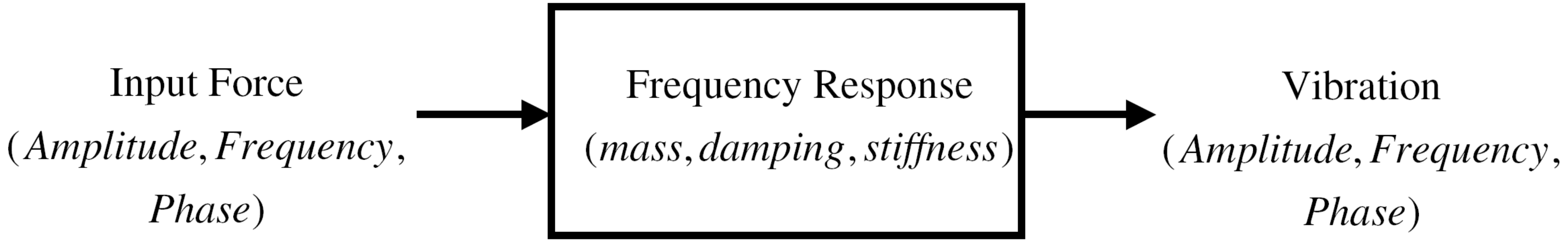


Frequence response function of a mass spring damper system

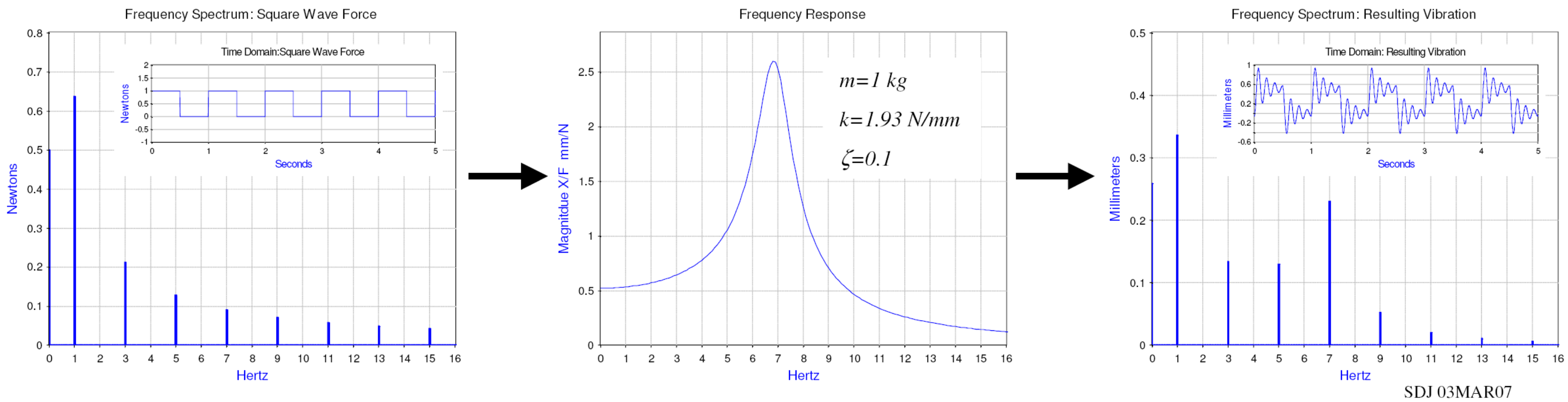


Logarithmic scales!

Input – FRF - Output



$$F(\omega) \quad x \quad H(\omega) \quad = \quad X(\omega)$$



Do not forget the phase change! Freq. dependent

Simulink Mass-Spring-Damper Example

Simulink example: MassaVeerDemperStep

Equivalent systems

Translational mechanics	Series RLC
Position x	Current i
Mass m	Inductance L
Spring k	Elastance 1/C
Damper b	Resistance R
Drive Force F(t)	di/dt

Series RLC:

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{L \cdot C}}$$

$$L \cdot \ddot{i} + R \cdot \dot{i} + i / C = \ddot{e}$$

Translational mechanics:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$m \cdot \ddot{x} + b \cdot \dot{x} + k \cdot x = F(t)$$

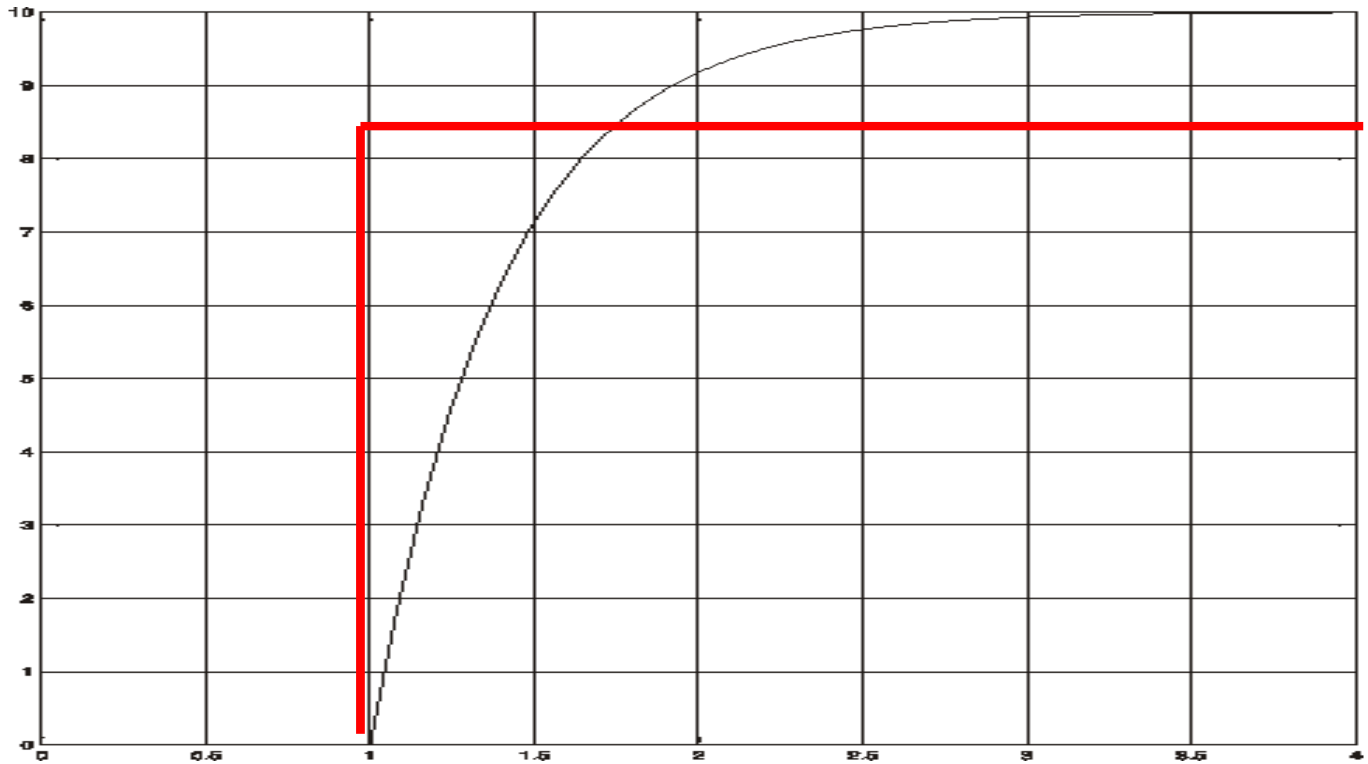
Realistic systems:

- Sum of N second order systems, $N \geq 0$
- Sum of M first order systems, $M \geq 0$
- Sum of P zero order systems, $P \geq 0$

The typical system responses, in the time domain, to an input step in the time domain.

SUMMARIES

1th order system response to step in time

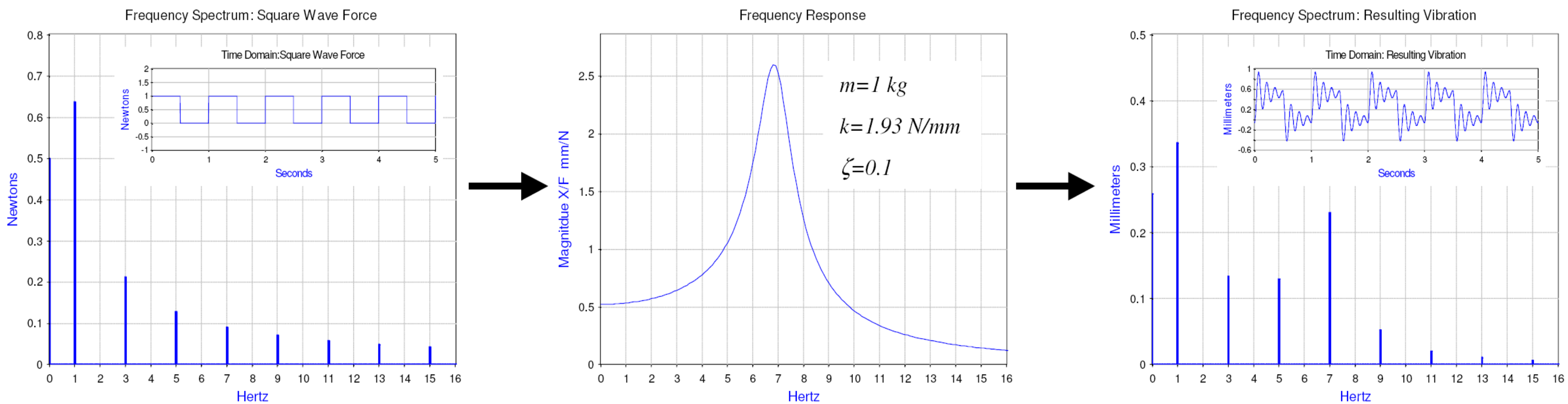


$$T(t) = T_{end} - (T_{end} - T_{begin}) \cdot e^{\frac{-t}{\tau}}$$

2^{nd} order system response to block function in time

Input Force
(*Amplitude*, *Frequency*, *Phase*)
→ **Frequency Response** (*mass*, *damping*, *stiffness*) →
Vibration
(*Amplitude*, *Frequency*, *Phase*)

$$F(\omega) \quad x \quad H(\omega) \quad = \quad X(\omega)$$



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