Finite Element Modelling Theory and application

Dr. ir. F.L.M. Delbressine

Literature

- Rao S.S., The Finite Element Method in Engineering, Elsevier, 2005.
- Zienkiewicz O.C, The Finite Element Method, McGraw-Hill, London, 1989.
- Przemieniecki J.S., Theory of Matrix Structural Analysis, McGraw-Hill, New York, 1968.

Basic idea behind FEM

- Find the solution of a complicated problem by replacing it by a simpler one

- Since the actual problem is replaced by a simpler one we will be able to find only an approximate solution!
- The existing mathematical models will not be sufficient to find the exact solution of most practical problems.

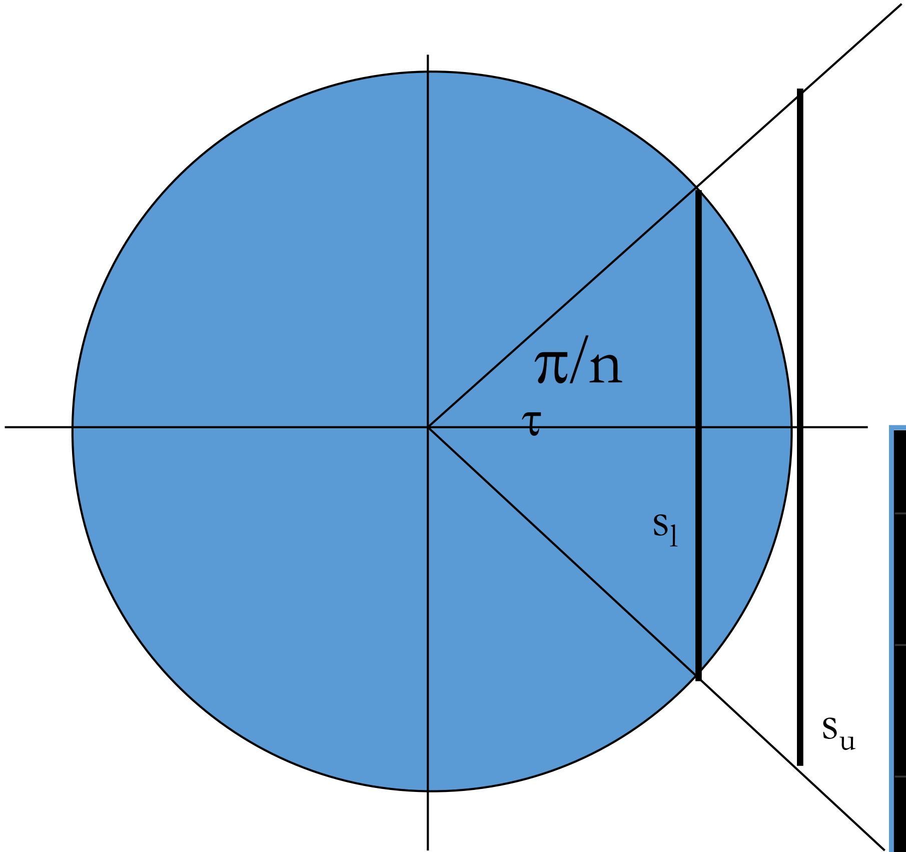
Finite Elements?

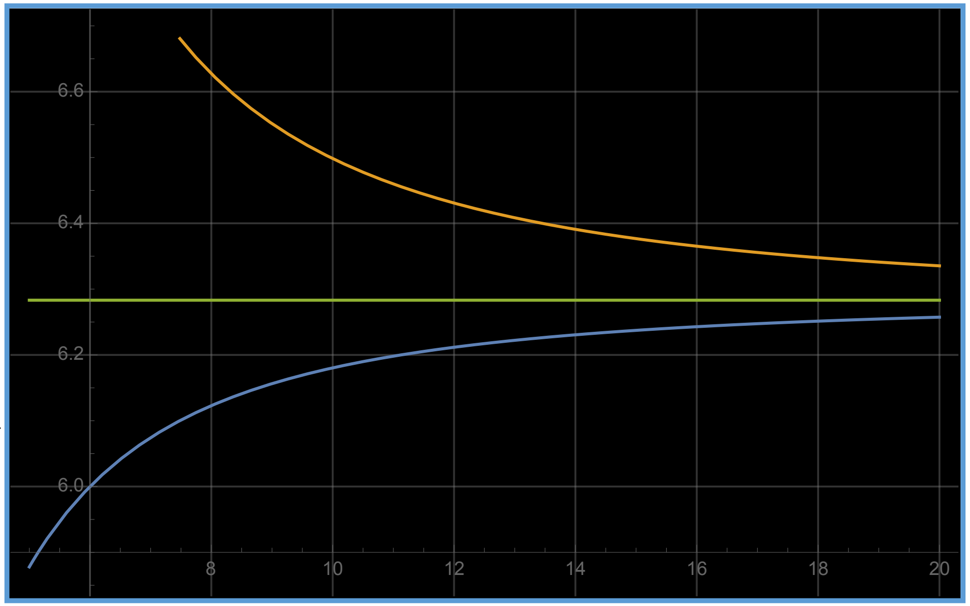
- The solution region is considered as built up of many small interconnected subregions called finite elements.
- In each element a convenient approximate solution is assumed and the conditions of overall equilibrium are derived.
- It will often be possible to improve or refine the approximate solution by spending more computational effort.

An approximation example

$$S_l = n \cdot s_l = 2 \cdot n \cdot R \cdot \sin(\frac{\pi}{n})$$

$$S_u = n \cdot s_u = 2 \cdot n \cdot R \cdot \tan(\frac{\pi}{n})$$





General applicability of the theory

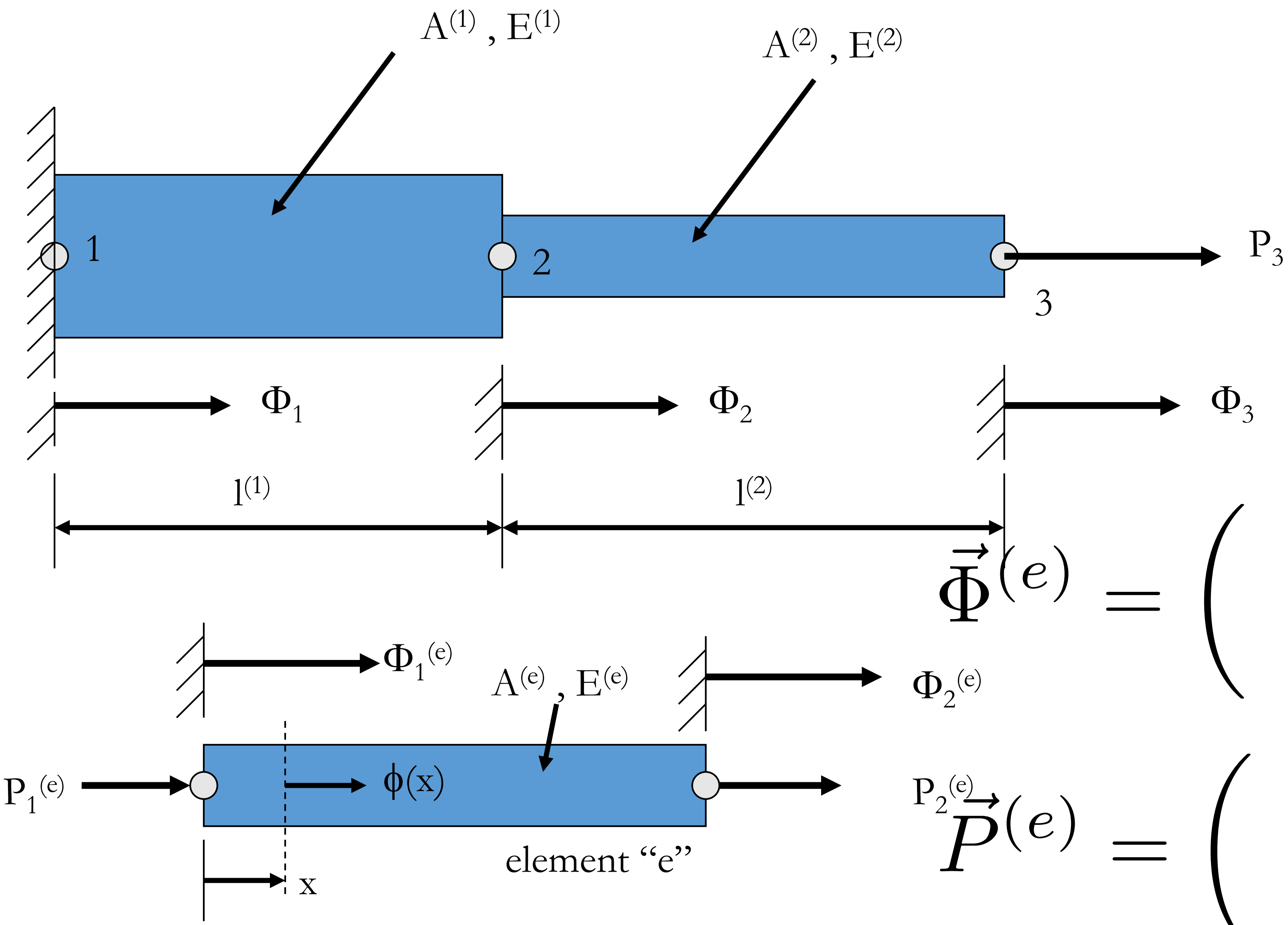
General description of the method

1. Discretization of the structure
2. Selection of a proper interpolation or displacement model
3. Derivation of element stiffness matrices and load vectors
4. Assemblage of element equations to obtain the overall equilibrium equations
5. Solution for the unknown nodal displacements
6. Computation of element strains and stresses

Example

stress analysis of a stepped bar

Discretization of the structure



$$\vec{\Phi}^{(e)} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}^{(e)}$$

$$\vec{P}^{(e)} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}^{(e)}$$

Selection of a proper interpolation or displacement model

$$\phi(x) = \Phi_1^{(e)} + \left(\Phi_2^{(e)} - \Phi_1^{(e)}\right) \cdot \frac{x}{l^{(e)}}$$

$$\Phi_1^{(e)} = \phi(0) \Rightarrow a = \Phi_1^{(e)}$$

$$\Phi_2^{(e)} = \phi(l^{(e)}) \Rightarrow b = (\Phi_2^{(e)} - \Phi_1^{(e)})/l^{(e)}$$

$$\phi(x) = \Phi_1^{(e)} + \left(\Phi_2^{(e)} - \Phi_1^{(e)}\right) \cdot \frac{x}{l^{(e)}}$$

Derivation of element stiffness matrices and load vectors

- Can be derived from the principle of minimum potential energy:

  I=strain energy –work done by external forces

$$I = \pi^{(1)} + \pi^{(2)} - W_p$$

$$\pi^{(e)} = A^{(e)} \cdot \int_0^{l^{(e)}} \frac{1}{2} \sigma^{(e)} \cdot \epsilon^{(e)} \cdot dx$$

$$\pi^{(e)} = \frac{A^{(e)} \cdot E^{(e)}}{2} \int_0^{l^{(e)}} {\epsilon^{(e)}}^2 dx$$

$$\epsilon^{(e)} = \frac{\partial \phi}{\partial x}$$

$$\epsilon^{(e)} = \frac{\partial \left(\Phi_1^{(e)} + \left(\Phi_2^{(e)} - \Phi_1^{(e)} \right) \cdot \frac{x}{l^{(e)}} \right)}{\partial x}$$

$$\epsilon^{(e)} = \frac{\left(\Phi_2^{(e)} - \Phi_1^{(e)} \right)}{l^{(e)}}$$

$$\pi^{(e)} = A^{(e)} \int_0^{l^{(e)}} \frac{1}{2}\sigma^{(e)} \cdot \epsilon^{(e)} \cdot dx$$

$$\sigma^{(e)} = E^{(e)} \cdot \epsilon^{(e)}$$

$$\pi^{(e)} = \frac{A^{(e)} \cdot E^{(e)}}{2} \int_0^{l^{(e)}} \epsilon^{(e)^2} \cdot dx$$

$$\pi^{(e)} = \frac{A^{(e)} \cdot E^{(e)}}{2} \int_0^{l^{(e)}} \left(\frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{l^{(e)}} \right)^2 dx$$

$$\pi^{(e)} = \frac{A^{(e)} \cdot E^{(e)}}{2} \left(\Phi_2^{(e)} - \Phi_1^{(e)} \right)^2$$

$$\pi^{(e)} = \frac{A^{(e)} \cdot E^{(e)}}{2} \left(\Phi_2^{(e)} - \Phi_1^{(e)} \right)^2$$

$$\pi^{(e)} = \vec{\Phi}^{(e)^T} \cdot K^{(e)} \cdot \vec{\Phi}^{(e)}$$

$$K^{(e)} = \frac{A^{(e)} \cdot E^{(e)}}{l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$W_p = \Phi_1 P_1 + \Phi_2 P_2 + \Phi_3 P_3 = \tilde{\Phi}^T \cdot \tilde{P}_C$$

$$\frac{\partial I}{\partial \Phi_i} = 0, i = 1, 2, 3$$

$$\frac{\partial I}{\partial \Phi_i} = \frac{\partial}{\partial \Phi_i} \left(\sum_{e=1}^{2} \pi^{(e)} - W_p \right) = 0$$

$$\sum_{e=1}^{2} \left([K^{(e)}] \cdot \vec{\Phi}^{(e)} - \vec{P}^{(e)} \right) = 0$$

$$[\tilde{K}] \cdot \vec{\tilde{\tilde{\Phi}}} - \vec{\tilde{\tilde{P}}} = \vec{0} \; \Rightarrow [\tilde{K}] \cdot \vec{\tilde{\tilde{\Phi}}} = \vec{\tilde{\tilde{P}}}$$

$$\Rightarrow \vec{\tilde{\tilde{\Phi}}} = [\tilde{K}]^{-1} \cdot \vec{\tilde{\tilde{P}}}$$

$$[\tilde{K}] \; = \sum_{e=1}^{2} [K^{(e)}]$$

$$\vec{\tilde{\tilde{\Phi}}} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} ; \vec{\tilde{\tilde{P}}} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

How to obtain the displacements given the loads and stifness matrix?

$$[\tilde{K}] = \begin{bmatrix} \frac{A^{(1)} \cdot E^{(1)}}{l^{(1)}} & -\frac{A^{(1)} \cdot E^{(1)}}{l^{(1)}} & 0 \\ -\frac{A^{(1)} \cdot E^{(1)}}{l^{(1)}} & \frac{A^{(1)} \cdot E^{(1)}}{l^{(1)}} + \frac{A^{(2)} \cdot E^{(2)}}{l^{(2)}} & -\frac{A^{(2)} \cdot E^{(2)}}{l^{(2)}} \\ 0 & -\frac{A^{(2)} \cdot E^{(2)}}{l^{(2)}} & \frac{A^{(2)} \cdot E^{(2)}}{l^{(2)}} \end{bmatrix}$$

$$\vec{\tilde{\Phi}} = [\tilde{K}]^{-1} \cdot \vec{\tilde{P}}$$

Computation of the element strains and stresses

$$\epsilon^{(1)} = \frac{\partial \phi}{\partial x} = \frac{\Phi_2^{(1)} - \Phi_1^{(1)}}{l^{(1)}} = \frac{\Phi_2 - \Phi_1}{l^{(1)}}$$

$$\epsilon^{(2)} = \frac{\partial \phi}{\partial x} = \frac{\Phi_2^{(2)} - \Phi_1^{(2)}}{l^{(2)}} = \frac{\Phi_3 - \Phi_2}{l^{(2)}}$$

$$\sigma^{(1)} = E^{(1)} \cdot \epsilon^{(1)} = E^{(1)} \frac{\Phi_2 - \Phi_1}{l^{(1)}}$$

$$\sigma^{(2)} = E^{(2)} \cdot \epsilon^{(2)} = E^{(2)} \frac{\Phi_3 - \Phi_2}{l^{(2)}}$$

The same example with numbers

$$A^{(1)} = 0.0002m^2$$

$$A^{(2)} = 0.0001m^2$$

$$l^{(1)} = l^{(2)} = 0.1m$$

$$E^{(1)} = E^{(2)} = 2 \cdot 10^{11} \frac{N}{m^2}$$

$$P_3 = 10^4 N; P_2 = 0N; \Phi_1 = 0m$$

$$P_1 =?; \Phi_2 =?; \Phi_3 =?$$

$$[\tilde{K}] = \left[\begin{array}{ccc} \frac{A^{(1)} \cdot E^{(1)}}{l^{(1)}} & -\frac{A^{(1)} \cdot E^{(1)}}{l^{(1)}} & 0 \\ -\frac{A^{(1)} \cdot E^{(1)}}{l^{(1)}} & \frac{A^{(1)} \cdot E^{(1)}}{l^{(1)}} + \frac{A^{(2)} \cdot E^{(2)}}{l^{(2)}} & -\frac{A^{(2)} \cdot E^{(2)}}{l^{(2)}} \\ 0 & -\frac{A^{(2)} \cdot E^{(2)}}{l^{(2)}} & \frac{A^{(2)} \cdot E^{(2)}}{l^{(2)}} \end{array} \right]$$

$$\left[\begin{array}{ccc} 4 \cdot 10^8 & -4 \cdot 10^8 & 0 \\ -4 \cdot 10^8 & 6 \cdot 10^8 & -2 \cdot 10^8 \\ 0 & -2 \cdot 10^8 & 2 \cdot 10^8 \end{array} \right] \cdot \left(\begin{array}{c} 0 \\ \Phi_2 \\ \Phi_3 \end{array} \right) = \left(\begin{array}{c} P_1 \\ 0 \\ 10^4 \end{array} \right)$$

Results:

$$\Phi_2 = 0.25 \cdot 10^{-4} m$$

$$\Phi_3 = 0.75 \cdot 10^{-4} m$$

$$P_1 = -1 * 10^4 N$$

Element stresses and strains:

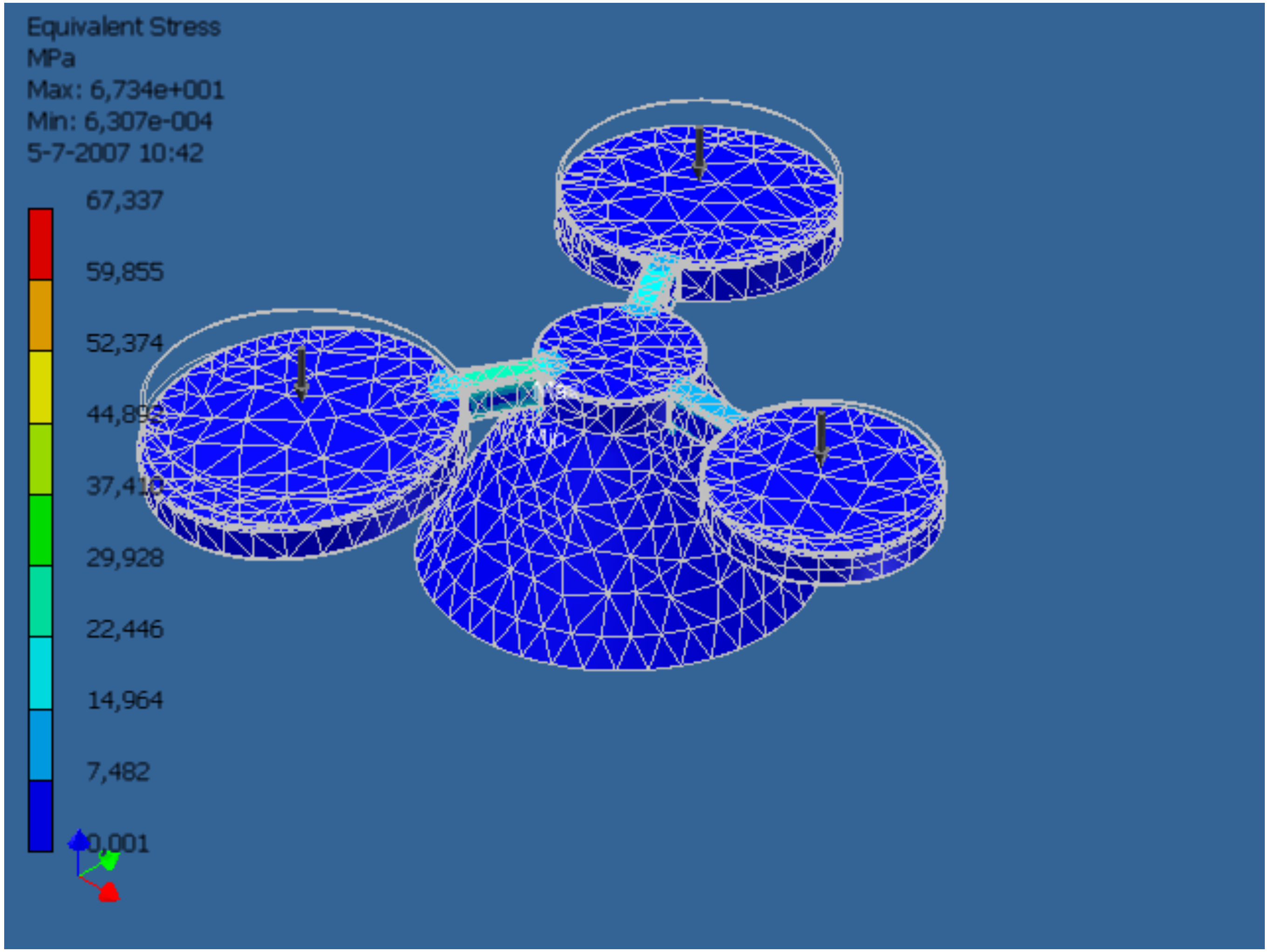
$$\epsilon^{(1)} = \frac{\Phi_2 - \Phi_1}{l^{(1)}} = 2.5 \cdot 10^{-4}$$

$$\epsilon^{(2)} = \frac{\Phi_3 - \Phi_2}{l^{(2)}} = 5 \cdot 10^{-4}$$

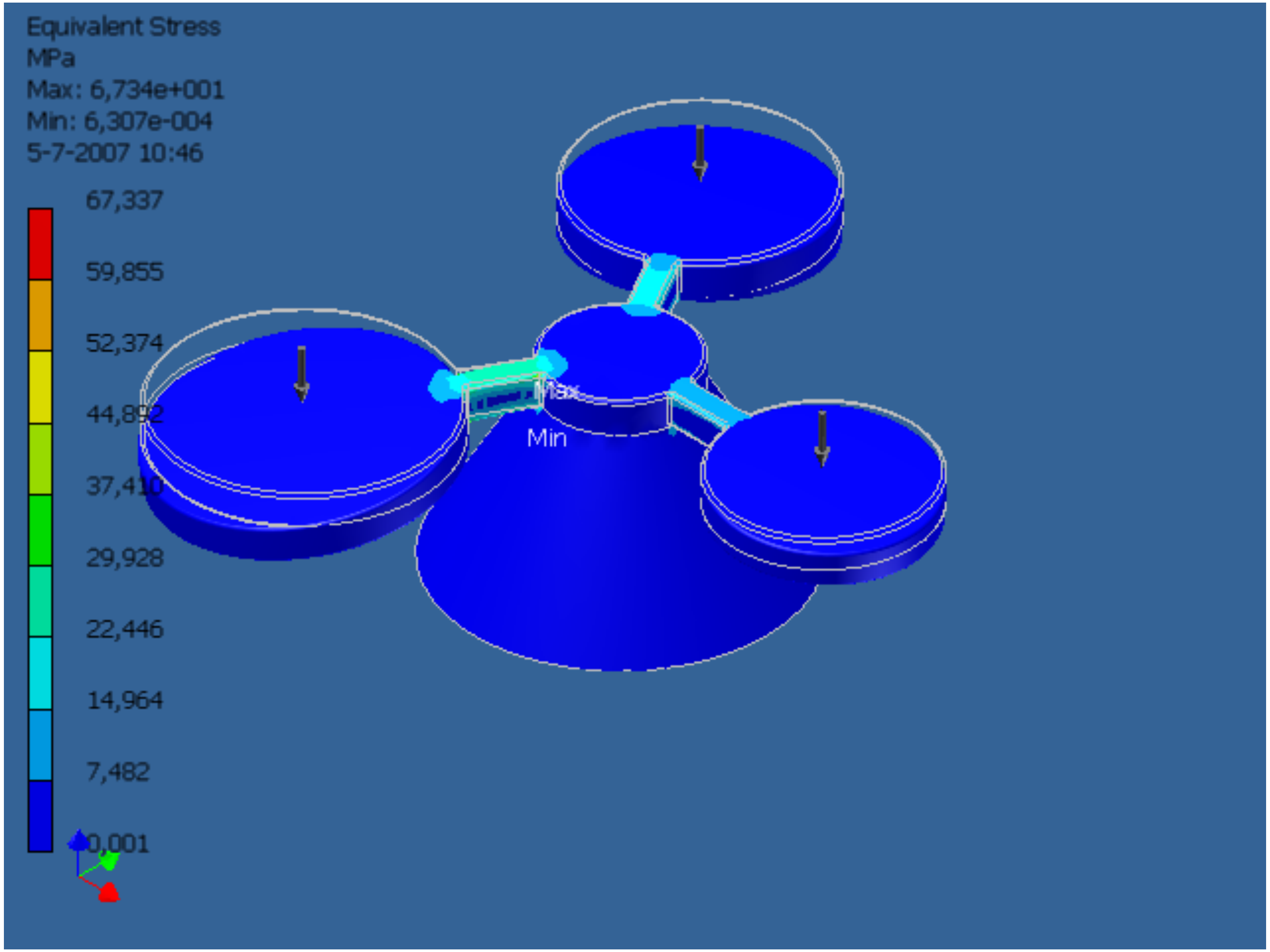
$$\sigma^{(1)} = E^{(1)} \cdot \epsilon^{(1)} = 5 \cdot 10^{7} \frac{N}{m^2}$$

$$\sigma^{(2)} = E^{(2)} \cdot \epsilon^{(2)} = 1 \cdot 10^{8} \frac{N}{m^2}$$

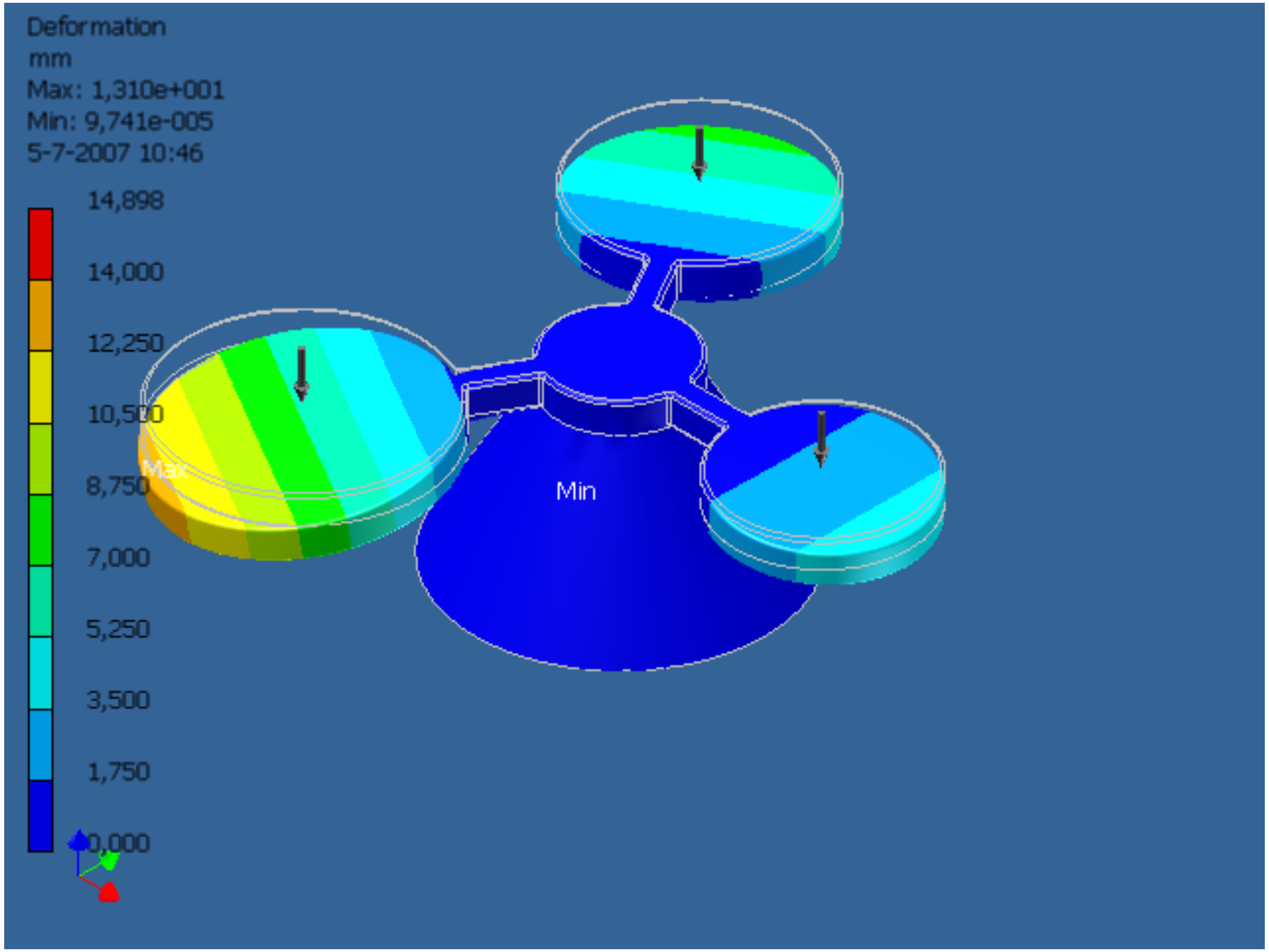
Example, Threechair, oak



Equivalent stress



Deformation



Safety factor