3.2 There Are Seven Types of Frieze Patterns

Figure 3.6 gives an example of a frieze pattern. This figure has the characteristic property that it is not invariant under arbitrary translations but only under powers of a "base translation" $\mathcal{T}$. Let us rephrase this as a mathemat-

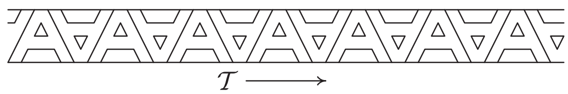

Fig. 3.6. A frieze pattern

ical definition. For a translation $\mathcal{T}$ and a positive integer k, we let $\mathcal{T}^k$ denote the k-fold composition of $\mathcal{T}$:

$$\mathcal{T}^k = \underbrace{\mathcal{T} \circ \mathcal{T} \circ \cdots \circ \mathcal{T}}_{k \text{ times}}.$$

Moreover, by convention: $\mathcal{T}^0 = \text{id}$ and $\mathcal{T}^{-k} = \left(\mathcal{T}^{-1}\right)^k$ for $k > 0$.

Definition 3.9. *A* frieze pattern *D is a subset of the plane for which the set of translations in the symmetry group $\mathcal{I}(D)$ is equal to $\{\mathcal{T}^k : k \in \mathbb{Z}\}$ for some translation $\mathcal{T}$. We say that $\mathcal{T}$* generates *the frieze pattern.*

If $\mathcal{T}$ generates the frieze pattern D, then so does $\mathcal{T}^{-1}$. A natural question is, how many different frieze patterns are there? From a practical point of view the answer is infinitely many. We can always slightly change the motif in a frieze pattern; Escher applied this principle often. If we want to answer the question from a mathematical point of view, we must first determine when two patterns are equivalent.

Definition 3.10. *We say that the frieze patterns D_1 and D_2* have the same type *if there is an isomorphism φ from $\mathcal{I}(D_1)$ to $\mathcal{I}(D_2)$ that maps $\mathcal{I}(D_1)^+$ onto $\mathcal{I}(D_2)^+$. We will say that two frieze patterns are* essentially different *if they do not have the same type.*

For example, the frieze patterns in Fig. 3.7 have the same type. For every pattern D in the figure, there is a translation $\mathcal{T}$ such that every translation that maps D onto itself is equal to $\mathcal{T}^k$ for some $k \in \mathbb{Z}$, while every translation $\mathcal{T}^k$, $k \in \mathbb{Z}$, maps D onto itself. We have

$$\mathcal{I}(D) = \mathcal{I}(D)^+ = \{\, \mathcal{T}^k : k \in \mathbb{Z} \,\}\ .$$

This group is isomorphic to $\mathbb{Z}$; to obtain an isomorphism we map $\mathcal{T}^k$ to k, for $k \in \mathbb{Z}$. The mathematical question is now, *how many essentially different frieze patterns are there*? We will show that there are seven essentially different frieze patterns.

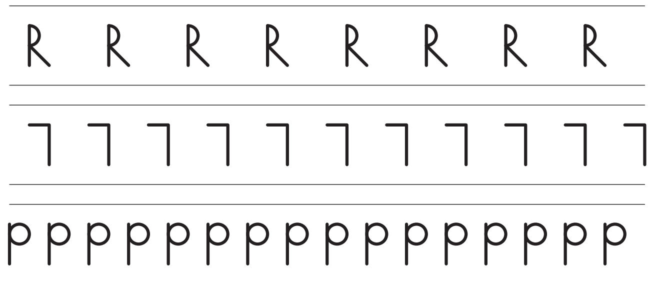

Fig. 3.7. Frieze patterns of the same type

We first list the different types of frieze patterns that have only direct isometries—either translations or rotations. The translations are all of the form $\mathcal{T}^k$, $k \in \mathbb{Z}$, where $\mathcal{T}$ is a translation that generates the frieze pattern. If a frieze pattern has rotational symmetry, this can only be twofold rotational symmetry.

Theorem 3.11. *Every rotation other than the identity in the symmetry group of a frieze pattern is of order* 2.

Proof. Let $\mathcal{T}$ be a translation that generates the frieze pattern. Let $\mathcal{R}$ be a rotation in the symmetry group, with center C. Consider the map $\mathcal{F} = \mathcal{R} \circ \mathcal{T} \circ \mathcal{R}^{-1}$. It follows from Sect. 2.4, Exercise 2.33 and Theorem 2.34 that $\mathcal{F}$ is a translation. We have

$$\mathcal{F}(C) = \mathcal{R} \circ \mathcal{T} \circ \mathcal{R}^{-1}(C) = \mathcal{R} \circ \mathcal{T}(C) \ .$$

Since $\mathcal{F} \in \mathcal{I}(D)$ and $\mathcal{F}$ is a translation, $\mathcal{F} = \mathcal{T}^k$ for some k, so

$$\mathcal{R} \circ \mathcal{T}(C) = \mathcal{F}(C) = \mathcal{T}^k(C) \ . \tag{3.1}$$

Since $\mathcal{R}$ is a rotation with center C, we obtain

$$d\left(C, \mathcal{T}^k(C)\right) = d\left(C, \mathcal{R} \circ \mathcal{T}(C)\right) = d\left(C, \mathcal{T}(C)\right) \ .$$

Consequently, $k = 1$ or $k = -1$. The first case, $k = 1$, does not occur. Indeed, (3.1) implies that in this case, $\mathcal{T}(C)$ is the rotation center of $\mathcal{R}$, whence $\mathcal{T}(C) = C$. This is impossible because a nontrivial translation has no fixed points. In the second case the rotation angle of $\mathcal{R}$ equals π, and $\mathcal{R}$ is of order 2.

We have found two types of frieze patterns, drawn on the left in Fig. 3.8. These can be seen as a succession of tiles. These tiles may be infinitely "high," but for practical reasons we will draw them with finite height. Together, the images of a single tile under the translations $\mathcal{T}^n$ cover the pattern, where the tiles have only boundary points in common. In the patterns on the left such a tile has been drawn. On the right the tile has been drawn again, now

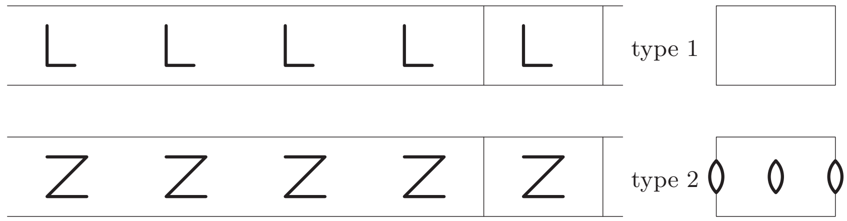

Fig. 3.8. Frieze patterns without indirect isometries

without motif. Instead, we have indicated the symmetry elements. The length of a tile, measured in the direction of the translation vector of the generating translation $\mathcal{T}$, is equal to the length of that vector.

A pattern of type 1 allows only translations. In the rest of this section we will denote the symmetry group of the frieze pattern of type 1 by H_1, and that of type 2 by H_2. The symmetry groups of frieze patterns are also called *frieze groups*. A pattern of type 2 allows both translations and rotations over π. The translations are of the form $\mathcal{T}^k$ with $k \in \mathbb{Z}$, where $\mathcal{T}$ is a generating translation. Note that $\mathcal{I}(D)$ being a group implies that as soon as there is one rotation $\mathcal{R}$ over an angle π that maps a frieze pattern D into itself, there are also other rotations with this property.

Let us examine this further. We find the rotations as follows; see Fig. 3.9. We write the rotation $\mathcal{R}$, with center C, as a product $\mathcal{S}_{l_2} \circ \mathcal{S}_{l_1}$ of two reflections

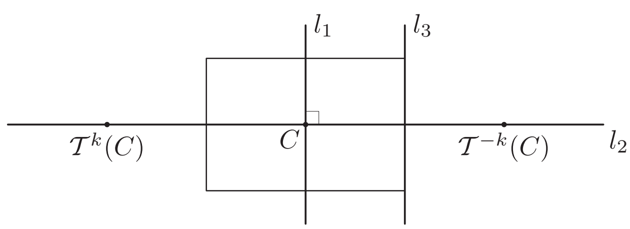

Fig. 3.9. $\mathcal{R} = \mathcal{S}_{l_2} \circ \mathcal{S}_{l_1}$, $\mathcal{T}^k = \mathcal{S}_{l_1} \circ \mathcal{S}_{l_3}$, $\mathcal{R} \circ \mathcal{T}^k = \mathcal{S}_{l_2} \circ \mathcal{S}_{l_3}$

with perpendicular axes l_1 and l_2, where l_2 is parallel to the span of $\mathcal{T}$, say $\mathcal{R} = \mathcal{S}_{l_2} \circ \mathcal{S}_{l_1}$. Next we write the translation $\mathcal{T}^k$, $k \neq 0$, as a product of two reflections with parallel axes l_1 and l_3 that are perpendicular to the span of $\mathcal{T}$, say $\mathcal{T}^k = \mathcal{S}_{l_1} \circ \mathcal{S}_{l_3}$; first $\mathcal{S}_{l_3}$, then $\mathcal{S}_{l_1}$. The product $\mathcal{R} \circ \mathcal{T}^k$ is the rotation $\mathcal{S}_{l_2} \circ \mathcal{S}_{l_3}$ with center the midpoint of the line segment $[C\mathcal{T}^{-k}(C)]$. From this we can easily deduce the rotation centers, indicated in Fig. 3.8 by small lenses. For later use we note that in the same way, using Exercise 2.31 it follows that

$$\mathcal{R} \circ \mathcal{T} = \mathcal{T}^{-1} \circ \mathcal{R} \quad \text{and} \quad \mathcal{R} \circ \mathcal{T}^{-1} = \mathcal{T} \circ \mathcal{R}\,. \tag{3.2}$$

Theorem 3.12. *Consider a frieze pattern D with generating translation $\mathcal{T}$ with translation vector $\mathbf{t}$. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are rotations in $\mathcal{I}(D)$ with distinct centers C_1 and C_2, respectively, there is a natural number k such that $d(C_1, C_2) = (k/2)\|\mathbf{t}\|$.*

Proof. The map $\mathcal{R}_1 \circ \mathcal{R}_2$ is a translation; the length of the translation vector is equal to $2d(C_1, C_2)$. This latter must be an integral multiple of $\|\mathbf{t}\|$. Therefore there is a natural number k such that $2d(C_1, C_2) = k\,\|\mathbf{t}\|$.

We continue the discussion we started before the theorem. We were studying which rotations can occur in the symmetry group of a pattern of type 2. We indicated these schematically in Fig. 3.8, in the lower right corner. The theorem we just proved implies that we have already found all rotations that can occur. Thus we arrive at the following classification of frieze patterns with only direct isometries.

Theorem 3.13. *If the symmetry group of a frieze pattern contains only direct isometries, it has type* 1 *or* 2*.*

Proof. Let $\mathcal{T}_i$, for $i = 1, 2$, be a generating translation of the frieze pattern of type i. In addition to translations, the symmetry group H_2 of the frieze pattern of type 2 also contains a rotation, which we denote by $\mathcal{R}_2$.

Above we saw that we can write every rotation in H_2 as a product $\mathcal{R}_2 \circ \mathcal{T}_2^k$ with $k \in \mathbb{Z}$. Let D be a frieze pattern with generating translation $\mathcal{T}$. If $\mathcal{I}(D)$ does not contain any rotations, $\mathcal{I}(D) = \{\, \mathcal{T}^k : k \in \mathbb{Z}\}$. By associating $\mathcal{T}^k$ to $\mathcal{T}_1^k$, for $k \in \mathbb{Z}$, we find an isomorphism from $\mathcal{I}(D)$ to H_1. If $\mathcal{I}(D)$ does contain a rotation, say $\mathcal{R}$, then in the same manner as above we find that the elements of $\mathcal{I}(D)$ are of the form $\mathcal{T}^k$ and $\mathcal{R} \circ \mathcal{T}^k$, $k \in \mathbb{Z}$. We define an isomorphism from $\mathcal{I}(D)$ to H_2 as follows: for every $k \in \mathbb{Z}$, we associate $\mathcal{T}^k$ to $\mathcal{T}_2^k$ and $\mathcal{R} \circ \mathcal{T}^k$ to $\mathcal{R}_2 \circ \mathcal{T}_2^k$. This map is indeed an isomorphism; to prove this we can use the equations in (3.2), which hold both in H_2 and in $\mathcal{I}(D)$. For example, the image of $\mathcal{R}_2 \circ \mathcal{T}_2^{-8} = \mathcal{T}_2^5 \circ (\mathcal{R}_2 \circ \mathcal{T}_2^{-3})$ is $\mathcal{T}^5 \circ (\mathcal{R} \circ \mathcal{T}^{-3}) = \mathcal{R} \circ \mathcal{T}^{-8}$.

To find the other types of frieze patterns, we will add indirect isometries to the symmetry groups H_1 and H_2. This method is the converse of what is described in Theorem 3.6. In that theorem we started out with a group of transformations and studied the place of the direct isometries in the group. Now we have a group H of direct isometries, where $H = H_1$ or $H = H_2$, and we try to expand this group to a group G in such a way that H is exactly the subgroup G^+. For every element $\mathcal{F}$ in the group G we want to construct, we have $\mathcal{F} \circ \mathcal{F} \in H$. This is because $\mathcal{F} \circ \mathcal{F}$ is always a direct isometry, regardless of whether $\mathcal{F}$ is direct. Our strategy is to repeatedly look for an indirect isometry $\mathcal{F}$ such that $\mathcal{F}^2$ is an element of H, and add this to H.

Extensions of H_1. Let us start with the symmetry group H_1. We have $H_1 = \{\, \mathcal{T}^k : k \in \mathbb{Z}\}$, where $\mathcal{T}$ is the generating translation of the type-1

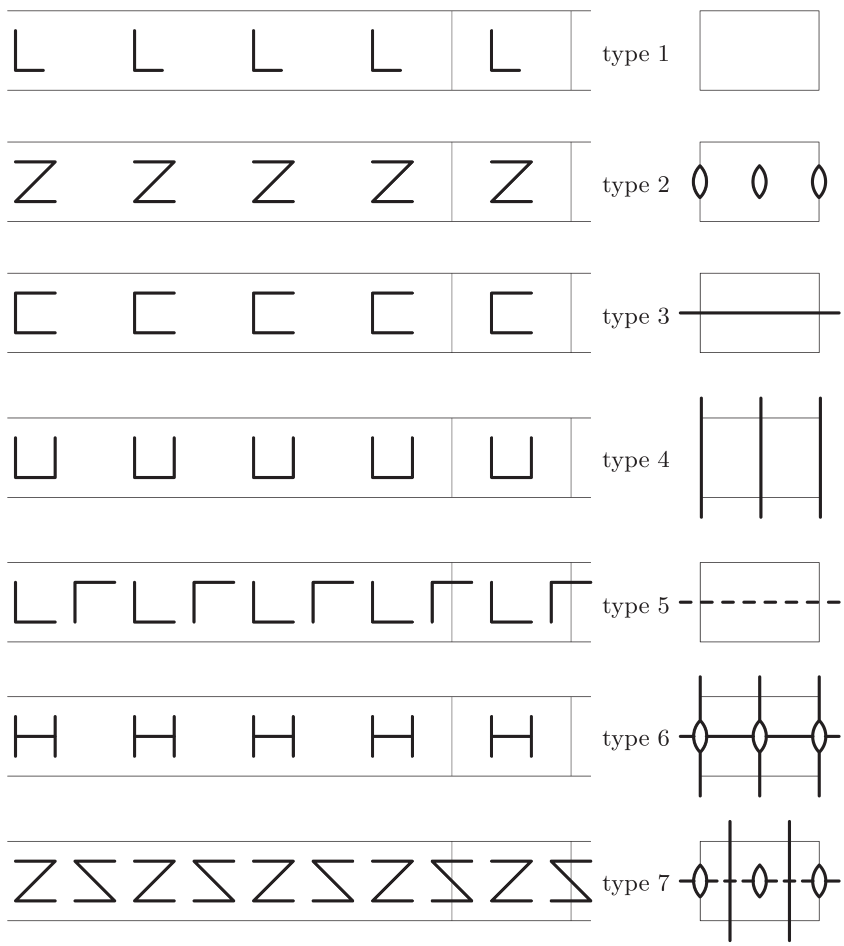

Fig. 3.10. The seven essentially different frieze patterns

frieze pattern. Which indirect isometries can we add to this? Since $\mathrm{id} = \mathcal{S}^2$ for every reflection $\mathcal{S}$, we first study which reflections we can add to H_1. The only reflections that qualify are those whose reflection axis is either perpendicular or parallel to the translation vector of $\mathcal{T}$; see Exercise 3.9. There can be only one of the latter type in the symmetry group, because otherwise we could find a composition that is a translation in a direction perpendicular to that of $\mathcal{T}$, which is excluded by the definition of a frieze pattern.

Let us first consider the extension of H_1 by a reflection with axis parallel to the translation vector of $\mathcal{T}$. This leads to the symmetry group H_3 of a type-3 frieze pattern; see Fig. 3.10. This has the following elements:

1. Translations $\mathcal{T}^n$, $n \in \mathbb{Z}$ (the elements of H_1);
2. A reflection $\mathcal{S}$ whose axis is parallel to the translation vector of $\mathcal{T}$;

3. Glide reflections, namely products of $\mathcal{S}$ and translations from H_1.

In the figure representing the tile, on the right, the reflection $\mathcal{S}$ is indicated by a continuous straight line. The symmetry group H_3 of the type-3 frieze pattern is one of the extensions of the translation group we had in mind.

We obtain a second extension of H_1 by adding a reflection whose axis is perpendicular to the translation vector of $\mathcal{T}$. The products of this reflection with the elements of H_1 are reflections. Figure 3.10 shows a type-4 frieze pattern. The symmetry group H_4 of this strip is exactly the extension of H_1 we just described. The representation of the tile, on the right, shows which reflections are in H_4.

There exists yet another extension of H_1. For this we write the translation $\mathcal{T}$ as the square of a glide reflection $\mathcal{G}$. The symmetry group H_5 of the type 5 frieze pattern contains $\mathcal{G}$, the translations, and other glide reflections. The square of such a glide reflection is a translation $\mathcal{T}^n$ with n odd, and every translation $\mathcal{T}^n$ with n odd can be obtained in this way. It follows that we have now found all extensions of H_1. The axis of a glide reflection is commonly indicated with a dashed line; see Fig. 3.10.

Extensions of H_2. We now turn our attention to extensions of the symmetry group H_2 of the type-2 frieze pattern; see Fig. 3.8. The symmetry group H_6 of the type-6 frieze pattern, Fig. 3.10, contains the elements of H_2 plus other reflections. Note that all these reflections can be obtained as product of one fixed reflection and the point reflections or translations from H_2.

The symmetry group H_7 of the type-7 frieze pattern contains both the elements of H_2 and a glide reflection $\mathcal{G}$ whose square is equal to $\mathcal{T}$. Since we are dealing with a group, we also find reflections in the symmetry group. We can, for example, find a reflection as shown in Fig. 3.11. The result is indicated

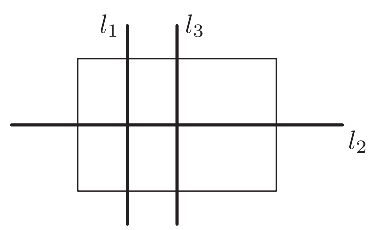

Fig. 3.11. $\mathcal{R} = \mathcal{S}_{l_3} \circ \mathcal{S}_{l_2}$, $\mathcal{G} = \mathcal{S}_{l_1} \circ \mathcal{S}_{l_2} \circ \mathcal{S}_{l_3}$, $\mathcal{G} \circ \mathcal{R} = \mathcal{S}_{l_1}$

schematically on the tile, on the right in Fig. 3.10; the dashed line indicates a glide reflection, the thick continuous lines indicate reflections. We have now found all possible extensions of H_2.

Theorem 3.14. *Every frieze pattern has one of the types* 1 *through* 7*.*

Proof. Let D be a frieze pattern. The group $\mathcal{I}(D)^+$ is isomorphic to either H_1 or H_2. If $\mathcal{I}(D) = \mathcal{I}(D)^+$, D must have type 1 or 2. If $\mathcal{I}(D) \neq \mathcal{I}(D)^+$, $\mathcal{I}(D)$ results from $\mathcal{I}(D)^+$ by adding a reflection or a glide reflection. The group $\mathcal{I}(D)$ therefore corresponds to one of the extensions of H_1 or H_2 described above. The proof of Theorem 3.13 shows how to make an isomorphism from $\mathcal{I}(D)$ to one of the symmetry groups H_3 through H_7.

In the following example we look back at Definition 3.10 of type. We will show why we need to bother with the direct isometries and may not simply say that D_1 and D_2 have the same type if there is an isomorphism from $\mathcal{I}(D_1)$ to $\mathcal{I}(D_2)$.

Example 3.15. We consider the symmetry group H_2 of the type-2 frieze pattern and the symmetry group H_4 of the type-4 frieze pattern. We will show that H_2 and H_4 are isomorphic. Nevertheless, types 2 and 4 are different, since $H_2^+ = H_2$ and $H_4^+ \neq H_4$.
We assume that the translations in both groups are powers of the same translation $\mathcal{T}$. We can choose coordinates in such a way that the centers of the rotations in H_2 are equal to $(n, 0)$, $n \in \mathbb{Z}$. We let $\mathcal{R}_n$ denote the rotation over π with center $(n, 0)$. Likewise, we can choose coordinates such that the axes of the reflections in H_4 have equations $x_1 = n$. We let $\mathcal{S}_n$ denote the reflection in the line $x_1 = n$. Note that in both cases, the "width" of the tile is equal to 2 and the norm of the translation vector is also equal to 2: $\|\mathcal{T}(\mathbf{o})\| = 2$. We can easily show that $\mathcal{R}_n = \mathcal{T}^n \circ \mathcal{R}_0$ and $\mathcal{S}_n = \mathcal{T}^n \circ \mathcal{S}_0$ for all integers n. To define the isomorphism from H_2 to H_4, we map $\mathcal{T}^n$ to $\mathcal{T}^n$ and $\mathcal{R}_n$ to $\mathcal{S}_n$, for all n; we can easily check that this defines an isomorphism.

Exercises

3.6. Show that the symmetry group of the type-1 and type-5 frieze patterns are isomorphic, whereas the types are different.

3.7. The symmetry groups of the type-2, -4, and -7 frieze patterns are isomorphic; nevertheless, the types are different.

3.8. Try to use the group properties to show that the symmetry groups of the type-3, -4, and -5 frieze patterns are not isomorphic.
Hint: The symmetry group of the type-5 frieze pattern contains no transformation whose square is the identity, other than the identity itself.

3.9. Let D be a frieze pattern with generating translation $\mathcal{T}$. If the reflection $\mathcal{S}$ is an element of the symmetry group $\mathcal{I}(D)$, the reflection axis of $\mathcal{S}$ is either parallel or perpendicular to the translation vector of $\mathcal{T}$.
Hint: Consider the map $\mathcal{S} \circ \mathcal{T} \circ \mathcal{S}$.