Plastic Number: Construction and Applications

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Abstract—**In this article we will construct plastic number in a heuristic way, explaining its relation to human perception in three-dimensional space through architectural style of Dom Hans van der Laan. Furthermore, it will be shown that the plastic number and the golden ratio special cases of more general definition. Finally, we will explain how van der Laan's discovery relates to perception in pitch space, how to define and tune Padovan intervals and, subsequently, how to construct chromatic scale temperament using the plastic number. (Abstract)**

Keywords-plastic number; Padovan sequence; golden ratio; music interval; music tuning ;

I. Introduction

In 1928, shortly after abandoning his architectural studies and becoming a novice monk, Hans van der Laan[1] discovered a new, unique system of architectural proportions. Its construction is completely based on a single irrational value which he called the *plastic*[2] *number* (also known as the *plastic constant*):

$$\Psi = 1.324718\ldots \approx \tfrac{4}{3}. \qquad (1)$$

This number was originally studied by G. Cordonnier[3] in 1924. However, Hans van der Laan was the first who explained how it relates to the human perception of differences in size between three-dimensional objects and demonstrated his discovery in (architectural) design. His main premise was that the plastic number ratio is "truly aesthetic in the original Greek sense, i.e. that its concern is not 'beauty' but *clarity* of perception" (see [1]).

In this section we will explain how van der Laan built his own system of proportions and in which way it determines all basic elements (building blocks) of his architectural style. In the following section we will compute (1) exactly and show that the plastic number and the golden ratio are two cases of the same definition, obtained by varying the single parameter, space dimension. In the final section plastic number is discussed within the pitch space of music. It will be explained how the plastic number is related to our perception of music intervals and to the Western 12-tone chromatic scale intonation (temperament).

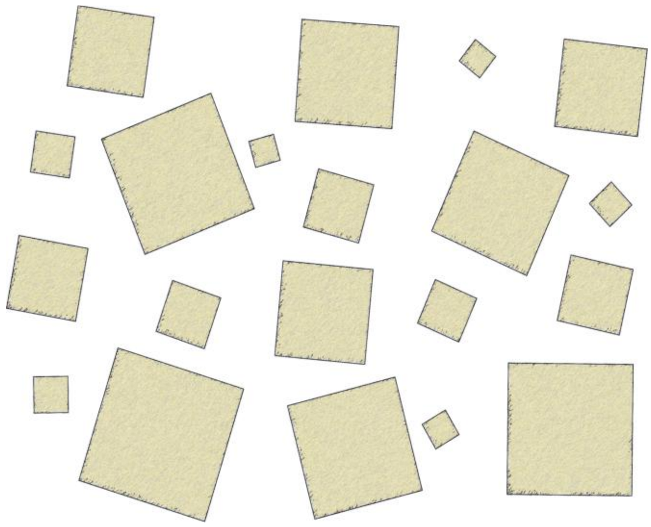

Figure 1. Twenty different cubes, from above

A. *Perception of Proportions: an Experimental Study*

Hans van der Laan thought that, perceiving a composition of objects in space, it only matters how their dimensions relate to each other, with the human body as a reference point. The following example illustrates his reasoning. Imagine a pile of, say, 20 wooden cubes. Let all cubes have a distinct size, where edge length of the smallest one equals 5 cm and edge length of the largest one 25 cm. Fig. 1 shows an example of such a set, viewed from above.

If cubes are close enough to each other, like they are in the picture, we automatically interpret them in terms of how they *relate* to each other. All cubes being of similar color, texture and shape, we obviously relate them in terms of *size*. Our brain will, furthermore, decide to relate *edge lengths*, since they represent, visually, the most acceptable information. Due to the limitations of our natural ability to measure and compute, these relations are interpreted as *ratios of small natural numbers*. This is what makes our perception so interesting.

In our example with set of cubes things are quite simple: we will automatically decide to relate different cubes in terms of their edge length. Once relations have been established, process of ordering them in a sequence naturally takes place. This is the most interesting part: how will our brain decide to accomplish such task? Indexing cubes with numbers 1-20 seems

1 Dom Hans van der Laan (1904-1991), was a Dutch architect and a member of the Benedictine Order.

2 "The word plastic was not intended to refer to a specific substance, but rather in its adjectival sense, meaning something that can be given a three-dimensional shape" (see [1]).

3 Gérard Cordonnier (1907-1977), was a French engineer. He studied the plastic number (which he called the *radiant number*) when he was just 17 years old.

too tedious to do at a glance. To construct shorter sequence we will first consider cubes of approximately the same size practically equal, thus creating groups and then ordering *them* by size. Van der Laan named that groups *types of size*. The reason for his scientific approach through an experimental study was to answer the following question: what is the smallest proportion $a : b$ of two lengths, where a is longer than b, by which they belong to different types of size?

Let us now imagine similar situation with 20 different cubes, but where edge length of largest cube equals 60 cm. Now the largest cube seems too big for the smallest one, because our brain considers relating them in terms of proportions too difficult and immediately ignores it. Such objects stay separated; when one is in focus of view, the other seems to disappear. This proves that types of size are related to each other up to some point where they differ too much. They are therefore grouped in categories, which van der Laan called *orders of size*, within which they can be easily related. The other question he wanted to answer empirically is: what is smallest proportion $b : a$ of two lengths, where a is longer than b, by which they belong to different orders of size?

Answers to these two questions precisely explain grounds of human perception, subsequently proving its objectivity. Hans van der Laan found these answers by conducting simple experiments in which objects had to be sorted by size. After statistical analysis, two numbers emerged: first of them, 4/3, answers the first question, and other one, 1/7, the second. We can express these important results in form of a single definition: *two objects belong to different types of size if quotient of their sizes is about 4/3 (plastic number), while they belong to different orders of size if one object is about seven times larger than the other.*

Ratio 1/7 is also used to determine whether two objects in the same type of size are in each other's neighborhood. Van der Laan used it in his work to relate building blocks to the openings between them. He calculated the ideal relation between the width of a space and the thickness of the walls that form it: seven to one on center. If it would exceed this ratio, the *nearness* of the elements – as van der Laan called it – would be lost.

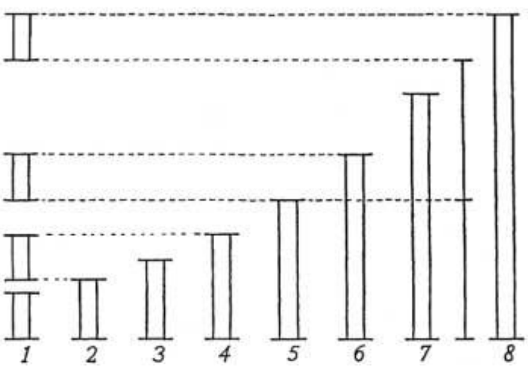

Figure 2. Types of size within an order of size (source: [1])

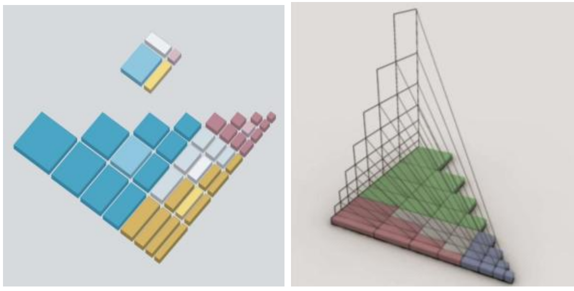

Figure 3. Construction of the Form Bank (source: [4])

System of proportions constructed from the plastic number is illustrated in Fig. 2. Lengths of bars shown in the picture increase by geometric progression, according to the plastic number Ψ, from 1 to $\Psi^7 = 7.1591\ldots \approx 7$. Therefore, they represent all eight types of size within one order of size. In the rest of this text, number $\frac{1}{\Psi^7} = 0.13698\ldots \approx \frac{1}{7}$ will be denoted by ψ.

B. *The Form Bank*

Hans van der Laan thought that the art of architecture lies in giving spaces meaningful differences, for which the plastic number is a unique tool. In his work, he distinguished three kinds of basic objects, or *forms*: blocks, bars, and slabs. He called their composition *thematismos*: the ordered arrangement of different forms. Using this, he created *the Form Bank*, a set consisting initially of 36 shallow base blocks, shown on the left side of Fig. 3. Their depth (height) is equal, while their widths and lengths increase according to the plastic number ratio, remaining in the same order of size. One group of blocks (red) has little difference in length, width and height. Another group, the bars (yellow), distinguish themselves with a variation in one direction. In the third group, the slabs (blue), measurements are extended in two directions relative to the smallest size. In the middle remains a fourth group (light gray) which van der Laan calls the *blanc shapes*. These last shapes have properties of all the aforementioned groups: block, bar and slab. In each group one "core" shape could be pointed out from the middle of each group (slightly lighter in color). These four powerful core shapes are the "representatives" of their group. On the right side of Fig. 3 base blocks are – in the same manner – varied in the third dimension, height, thus forming a quasi-tetrahedron, which completes the construction of the Form Bank. Its elements are illustrated in Fig. 4.

Hans Van der Laan's most famous work is the Abbey of St. Benedictusberg in Vaals (see Fig. 5), consisting of an upper church, a crypt and an atrium (1956-1968). He built several other monasteries (Abbey Roosenberg Waasmunster, Monastery Mary sisters of St. Francis, Monastery church Tomelilla) and also a private house "Huis Naalden". He not only designed buildings, but also furniture and even a typeface (see Fig. 6). He chose the heights of a bench and a chair to be the starting point and then applied the system of measurements of the plastic number to create the other dimensions. The typeface he invented, *the Alphabet in stone*, is based on the Roman carved stone capitals. The basic figures of the letters are the square and the rectangle in ratio four to three, which then form the letters and even the spaces between them.

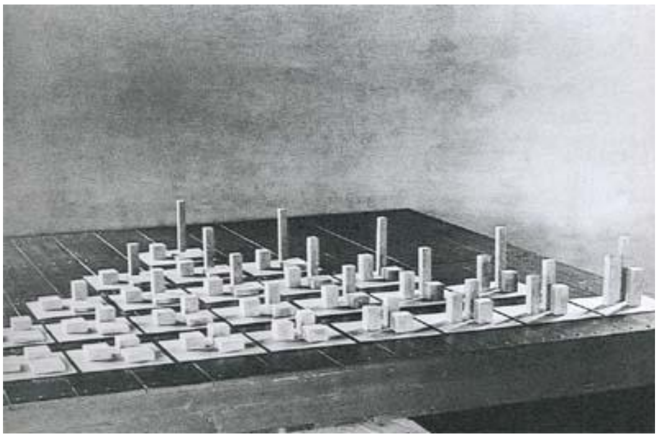

Figure **4.** Elements of the Form Bank

Figure 5. The Abbey in Vaals (Frans de la Cousine)

Figure 6. Van der Laan's typeface and furniture design

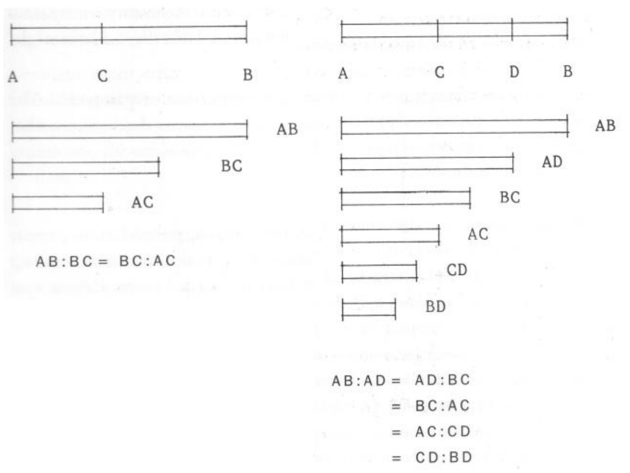

Figure **7.** Breaking the segment AB in 2 and 3 parts (source: [2])

II. Construction of the Plastic Number

A. *Evolution of the Golden Ratio*

Hans van der Laan concluded that number 4/3 is the rational approximation of some value which should be precisely definable. The base of his mathematical research is one possible construction of number Φ, called *golden ratio* or *divine proportion*, an ancient aesthetical axiom. The latter works well in plane (its simplest representation being the golden rectangle), but fails to generate harmonious relations within and between *three-dimensional* objects. Van der Laan therefore elevates definition of the golden rectangle in terms of space dimension.

Golden ratio can be calculated by sectioning the segment *AB* in two parts *AC* and *BC* such that

$$\Phi = \frac{AB}{BC} = \frac{BC}{AC}, \tag{2}$$

as shown on the left side of Fig. 7. Segments *AB* and *BC* are sides of the golden rectangle. Letting $AB = 1$ it follows

$$\Phi = \frac{1}{BC} = \frac{BC}{1-BC} \Rightarrow BC^2 = 1 - BC \Rightarrow \Phi^2 = \Phi + 1. \tag{3}$$

Golden ratio is obtained by solving last equation in (3):

$$\Phi = 1.618034\ldots \tag{4}$$

Van der Laan breaks segment *AB* in the similar manner, but in *three* parts. If *C* and *D* are points of subdivision, plastic number Ψ is defined with

$$\Psi = \frac{AB}{AD} = \frac{AD}{BC} = \frac{BC}{AC} = \frac{AC}{CD} = \frac{CD}{BD}, \tag{5}$$

as illustrated on the right side of Fig. 7. Letting $AB = 1$, from $AC = 1 - BC$, $BD = 1 - AD$ and (5) follows

$$\Psi^3 = \Psi + 1. \tag{6}$$

Using Cardano's formula, (1) is obtained from (6) as the only real solution. Segments *AC*, *CD* and *BD* can be interpreted as sides of a cuboid analogous to the golden rectangle. That cuboid is obviously contained in the Form Bank (light blue rectangle in Fig. 3 is its base).

B. Harmonious numbers

Numbers Ψ and Φ can be obtained from more general definition, which is based on the following two theorems.

Theorem 1. *For given natural number* $n \ge 2$, *let*

$$f_n(x) = x^n - x - 1. \tag{7}$$

Then there exists real number $\chi_n \in (1,2)$ *such that* $f_n(\chi_n) = 0$ *and for every root* $r \in \mathbf{C}$ *of the polynomial* f_n *the following statement holds:*

$$r \ne \chi_n \quad \Rightarrow \quad |r| < \chi_n. \tag{8}$$

<u>*Proof sketch.*</u> Condition $n \ge 2$ implies that $P_n(1) < 0$ and $P_n(2) > 0$, therefore existence of number χ_n follows from the intermediate value theorem. The rest of the statement follows mainly from Eneström-Kakeya theorem (see [9]). □

For $n = 2$ or 3, (7) matches (3) or (6), respectively; therefore it is $\chi_2 = \Phi$ and $\chi_3 = \Psi$.

Corollary 1. *For any natural number* $n \ge 2$ *is* $\chi_{n+1} < \chi_n$.

<u>*Proof.*</u> Assume the contrary, i.e. that there exists natural number $n \ge 2$ such that $\chi_{n+1} \ge \chi_n$. Then (7) implies

$$\chi_{n+1}^n \ge \chi_n^n \Rightarrow \frac{1}{\chi_{n+1}} + 1 \ge \chi_n + 1 \Rightarrow \chi_n \cdot \chi_{n+1} \le 1 \tag{9}$$

which is impossible. Therefore $\chi_{n+1} < \chi_n$ for all $n \ge 2$. □

Theorem 2. *For given natural number* $n \ge 2$, *let*

$$H_k^{(n)} = \begin{cases} 1, & k = 1,2,\ldots,n, \\ H_{k-n}^{(n)} + H_{k-n+1}^{(n)}, & k > n. \end{cases} \tag{10}$$

Then number χ_n *from Theorem 1 is defined with*

$$\chi_n = \lim_{k\to\infty} \frac{H_{k+1}^{(n)}}{H_k^{(n)}}. \tag{11}$$

<u>*Proof sketch.*</u> It is easy to see that (7) is characteristic polynomial for recurrence relation (10). Hence, there exist numbers $C_j \in \mathbf{C}$, $j = 1,2,\ldots,n$, such that $H_k^{(n)} = \sum_{j=1}^{n-1} C_j r_j^k + C_n \chi_n^k$, where $r_1, r_2, \ldots, r_{n-1} \in \mathbf{C}$ are roots of (7) different than χ_n. Now it is not difficult to prove that (8) implies (11). □

Sequence $\left(H_k^{(n)}\right)_k$ is Fibonacci sequence for $n = 2$ (see [7]) and Padovan sequence for $n = 3$ (see [6]).

Let now $K_k^{(n)}$ be a hyperrectangle in *n*-dimensional space, $n \ge 2$, defined as Cartesian product of *n* intervals:

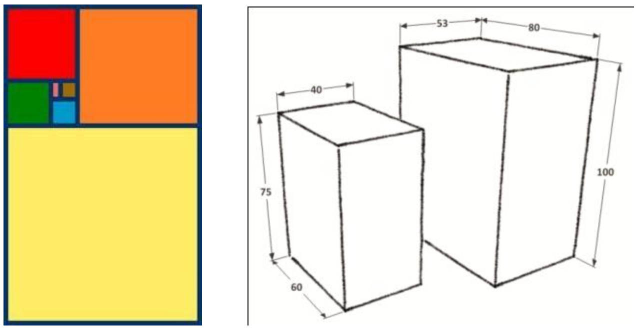

Figure 8. Harmonious number ratio in the plane (left) and space (right)

$$K_k^{(n)} = \prod_{j=0}^{n-1} [0, H_{k+j}^{(n)}]. \tag{12}$$

Length of the longest side of $K_{k+1}^{(n)}$ equals the sum of two shortest sides of $K_k^{(n)}$. Now, (11) can be interpreted as the limit of quotient of longest sides of $K_{k+1}^{(n)}$ and $K_k^{(n)}$ when $k \to \infty$. For large *k* these two adjacent hyperrectangles are considered almost similar in shape while ratio of their sizes equals approximately χ_n. The latter is called *harmonious number* (or *clarity ratio*) for *n*-dimensional Euclidean space. That is because up to the limit of our perception of space ($n \le 3$), harmonious number represents smallest (greater than 1) ratio of sizes of two similar *n*-dimensional objects such that

a) their difference in size is immediately noticeable,

b) relation between their sizes is perceived as clear, pleasant, harmonious, natural and stable.

Statements a) and b), demonstrated for $n = 2$ and $n = 3$ in Fig. 8, are in fact nonsensical for $n > 3$ because of perception limits[4]; concept of those harmonious numbers is just an interesting inductive hypothesis. Approximate values of the first seven harmonious numbers are given in the following table.

n	2	3	4	5	6	7	8
χ_n	1.6180	1.3247	1.2207	1.1673	1.1347	1.1128	1.0970

First two harmonious numbers stand apart being only ones we can *experience*. Moreover, they differ from others in pure mathematical, abstract sense. One could easily prove that for $x = \Psi$ or $x = \Phi$ there exist *natural* numbers *p* and *q* such that the following is satisfied:

$$x + 1 = x^p, \quad x - 1 = x^{-q}. \tag{13}$$

Real numbers $x > 1$ that satisfy these conditions are called *morphic numbers*. It can be shown (see [5]) that the plastic number and the golden ratio are the only two such numbers. Because $\chi_n > 1$ for every $n \ge 2$, these are only harmonious morphic numbers.

[4] Perceiving the size of an object by viewing implies relating lengths of *all* its sides (to determine the longest one). Hence one must perceive these first, what is possible, obviously, only in three-dimensional space and its subspaces.

III. PLASTIC NUMBER IN MUSIC

A. Music Intervals

In this section we will establish connection between the plastic number and the standard Western 12-tone music system (chromatic scale). Let us define intervals within the chromatic scale, spanning one octave, as mathematical objects.

Since intervals re measured in semitones and there are 12 semitones between root tones of two adjacent scale instances, we typically represent them with elements of cyclic group $\mathbf{Z}_{12}$. It contains whole numbers from 0 to 11, which are mapped to set of 12 basic intervals within an octave (unison, minor second, major second, minor third, major third, perfect fourth, tritone, perfect fifth, minor sixth, major sixth, minor seventh and major seventh) in one-to-one fashion, being their width in semitones. Furthermore, standard addition of music intervals, defined as sum of their widths, matches addition in $\mathbf{Z}_{12}$ since sum of intervals which exceeds octave represents tone in next scale instance, so it can be reduced to basic interval by subtracting 12 semitones.

We will denote group of basic intervals with $\mathbf{I} = \mathbf{Z}_{12}$. To keep further notation simple, let us observe that for every whole number n there is a unique interval m in $\mathbf{I}$ such that n is congruent to m modulo 12. We denote $\overline{n} = m$, thus defining map

$$n \in \mathbf{Z} \to \overline{n} \in \mathbf{I}. \qquad (14)$$

B. The Pitch Space

When two tones T_1 and T_2 are sounding simultaneously, we automatically relate their *fundamental frequencies* φ_1 and φ_2 (in further text *frequencies*). We therefore observe them in pitch space, which comprises all fundamental frequencies. Frequency ratio $\varphi_2 : \varphi_1 \geq 1$ is called an *interval* from T_1 (lower tone) to T_2 (higher tone).

Our brain interprets frequencies as their logarithms, so T_1 and T_2 are perceived as $\log(\varphi_1)$ and $\log(\varphi_2)$, while interval between them is $\log(\varphi_1 : \varphi_2) = \log(\varphi_1) - \log(\varphi_2)$. It seems to us that interval between T_1 and T_2 is *differentiation* $T_1 - T_2$.

The following phenomenon occurs: two frequency ratios r_1 and r_2 are perceived as the same interval in different octaves if $r_1 = r_2 \cdot 2^n$ for some whole number n. This is an equivalence relation: we will denote $r_1 \sim r_2$. Every frequency ratio r therefore has its unique *representant* $\overline{r} \in [1,2)$ such that $r \sim \overline{r}$, defined with

$$\overline{r} = r \cdot 2^{-\lfloor \log_2 r \rfloor}. \qquad (15)$$

To "materialize" the 12-tone chromatic scale in pitch space, i.e. to translate written music to sound, we must define frequency ratio $\tau_i \in [1,2)$, which we call *tuning* or *intonation*, for every basic interval i in $\mathbf{I}$. That is called *scale temperament*. Example is the equal temperament

$$i \to \tau_i^{EQ} = \sqrt[12]{2^i}, \quad i \in \mathbf{I}. \qquad (16)$$

Given the tone of frequency φ and the whole number n, the set of frequencies $\{2^n \cdot \varphi \cdot \tau_i : i \in \mathbf{I}\}$ is tone representation of the chromatic scale in n-th octave of the pitch space. Observe that $\overline{i+j} = l$, where i, j and l are basic intervals, implies that it has to be at least $\overline{\tau_i \cdot \tau_j} \approx \tau_l$, otherwise the basic music theory would cease to make sense[5]. How precise must the latter approximation be? Perceived difference (interval) between tones of frequencies φ_1 and φ_2 is typically defined with $|d(\varphi_1, \varphi_2)|$, where

$$d(\varphi_1, \varphi_2) = 1200 \cdot \log_2 \frac{\varphi_1}{\varphi_2} \text{ [cents]}. \qquad (17)$$

One octave is therefore 1200 cents wide. Smallest audible difference between two frequencies is about 10 cents. Minor second, smallest interval within the chromatic scale, is 100 cents wide. Therefore, two tunings τ_i and υ_i of the same interval i, belonging to different temperaments, should differ by less than quarter-tone, i.e. satisfy $|d(\tau_i, \upsilon_i)| < 50$.

C. Types and Orders of Size in Pitch Space

While experimenting with monochord, Pythagoras concluded that intervals with simplest frequency ratios sound most harmonious[6]. These are ratios 2 : 1, 3 : 2 and 4 : 3, which represent only three perfect intervals: octave, fifth and fourth, respectively. At the time when polyphonic music of Europe started to develop these were only consonant intervals; all others were considered dissonant (unstable). Because Ψ is slightly smaller than 4/3, more precisely $d\left(\frac{4}{3}, \Psi\right) \approx 11.22$ cents, it can be interpreted as the highest lower bound for tunings of perfect intervals. In other words, tuning Ψ resembles first perfect interval after trivial unison. Therefore, for the given tone of frequency φ, set of frequencies $\{\varphi \cdot \Psi^{n-1} : n = 1, 2, \ldots, 8\}$ is an order of size, while its elements are types of size in pitch space. Its width is $\Delta = \frac{\varphi \cdot \Psi^7}{\varphi} = \Psi^7 \approx 7.1592$, which is slightly wider than two octaves plus minor seventh: $d(\overline{\Delta}, t_{10}^{EQ}) \approx 7.76$ cents. Therefore, Δ embraces first 7 elements of harmonic series (fundamental tone and first 6 overtones). If the distance between two tones belonging to instances of chromatic scale is less than Δ, it is easy to relate them, because spectra of both tones are audible enough on the intersection of their domains. When the distance is larger than Δ, relating becomes difficult, especially for dis-

[5] This is the main problem of scale temperation. For chromatic 12-tone scale it has several solutions, which were given in the course of centuries. Finally, the equal temperament was chosen as the least compromising one.

[6] An interval between two tones seems as harmonious as audible intersection of their spectra is, i.e. how soon their higher harmonics (overtones) start to coincide.

sonant intervals; when it spans more than several octaves, quality of consonance, or harmony, is also lost – all such intervals sound more or less dissonant.

Perfect octave, or frequency ratio 2 : 1, satisfies conditions a) and b). It is hence harmonious number for the pitch space, which is one-dimensional by the definition. Letting $\chi_1 := 2$, previous definition of harmonious numbers extends to the whole category of Euclidean spaces $\{E_n : n \in \mathbf{N}\}$.

D. *Plastic Number Temperament*

Finally, it will be shown how van der Laan's types of size $S(n) = \Psi^{n-1}$, where n is natural number, generate possible tunings for basic intervals. Multiplying (7) by Ψ^{n-1} yields

$$\Psi^{n+2} = \Psi^n + \Psi^{n-1} \quad \Rightarrow \quad S(n+3) = S(n+1) + S(n) \qquad (18)$$

for every natural number n; hence types of size satisfy Padovan recurrence. Members of the sequence $(p_n)_n = (P_{n+4})_n$, where P_n is n-th Padovan number, obviously have the same property. They form the largest subsequence of the Padovan sequence such that all its members are mutually different:

$$(p_n)_n = 2,3,4,5,7,9,12,16,21,28,37,49,\ldots \qquad (19)$$

Interval which spans p_n semitones is called Padovan interval and is denoted simply by p_n (first ten of them are shown in Fig. 9). Now, it is natural to assume that $S(n)$ is proportional to the logarithm of frequency ratio r_n of interval p_n, i.e.

$$\log_2(r_n) = \lambda \cdot S(n). \qquad (20)$$

Because $p_7 = 12$, r_7 should equal 2; hence $\lambda = 1/S(7)$.

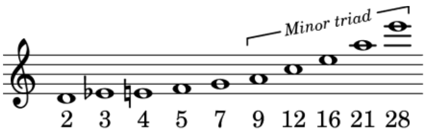

Figure **9**. First ten Padovan intervals, built on the middle *C*

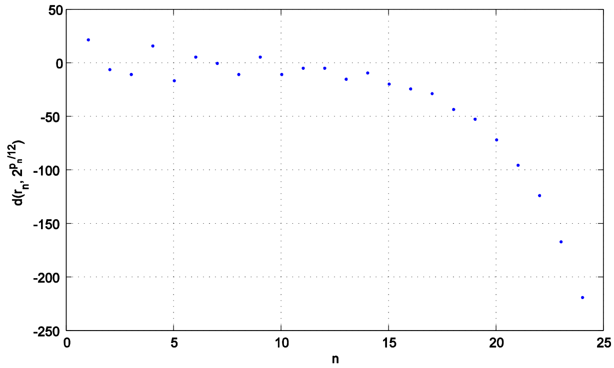

Figure 10. Deviation of plastic number tunings from equal temperament

TABLE 1. PLASTIC NUMBER TEMPERAMENT

Interval	Ratio	Deviation	Interval	Ratio	Deviation
Unison	1.0000	0	Tritone	1.3985	–19.31
Min. 2nd	1.0567	–4.49	Perf. 5th	1.5111	+14.82
Maj. 2nd	1.1167	–8.97	Min. 6th	1.5969	+10.34
Min. 3rd	1.1852	–5.85	Maj. 6th	1.6875	+5.85
Maj. 3rd	1.2524	–10.34	Min. 7th	1.7910	+8.97
Perf. 4th	1.3235	–14.82	Maj. 7th	1.8927	+4.49

Solving (20) $r_n = 2^{\psi \cdot \Psi^n}$ is obtained. Its representant tuning $\bar{r}_n$ is given with

$$\bar{r}_n = 2^{\psi \cdot \Psi^n} \cdot 2^{-\lfloor \log_2 \psi \cdot \Psi^n \rfloor} = 2^{\psi \cdot \Psi^n - \lfloor \psi \cdot \Psi^n \rfloor} = 2^{\{\psi \cdot \Psi^n\}} \qquad (21)$$

where $\{x\} = x - \lfloor x \rfloor$ denotes *fractional part* of some real number $x > 0$. These tunings can be used to build original plastic number temperament shown in Table 1, as follows.

Fig. 10 shows deviation of frequency ratios r_n from $2^{p_n/12}$, the latter being equally tempered intervals p_n, as they are in the modern piano. When $n > 15$, that difference starts to grow rapidly. Therefore, only r_n for $n \le 15$ are usable. Indexes n of the most suitable tunings for first seven basic intervals are chosen from the following table.

n	$\bar{p}_n$	$d(\bar{r}_n, \tau^{EQ}_{\bar{p}_n})$	n	$\bar{p}_n$	$d(\bar{r}_n, \tau^{EQ}_{\bar{p}_n})$	n	$\bar{p}_n$	$d(\bar{r}_n, \tau^{EQ}_{\bar{p}_n})$
1	2	+22.04	6	9	+5.85	11	1	–4.49
2	3	–5.85	7	0	0	12	1	–4.49
3	4	–10.34	8	4	–10.34	13	5	–14.82
4	5	+16.19	9	9	+5.85	14	2	–8.97
5	7	–16.19	10	4	–10.34	15	6	–19.31

First seven frequency ratios (from unison to tritone) in Table 1 are hence obtained by computing $\bar{r}_n$ for $n = 7,11,14,2,3,13$ and 15, respectively. Others are computed using the principle of the inverse interval, i.e. dividing 2 by tuning for minor second yields tuning for major seventh and so forth.

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