Exploring a Taxicab-based Mandelbrot-like Set

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Abstract

We describe the development and application of a novel fractal. The fractal resembles the well-known Mandelbrot fractal, but it has different aesthetic qualities. Whereas the traditional quadratic mapping uses the addition and multiplication of complex numbers, for this work, we define an unconventional multiplication operation, inspired by the Taxicab metric. We explore the visual features of the fractal, aiming to design an attractive fashion item.

Introduction

We describe the development and application of a novel fractal, resembling the Mandelbrot fractal [9], but having different aesthetic qualities. In earlier projects, we took inspiration from weaving techniques, and by processes of formalization, creative programming, and garment design we arrived at innovative fashion pieces. The goal of the research described in this paper was to see whether we could create a kind of mixling by interbreeding between the orthogonal structure underlying all weaving and the mathematical principles of the Mandelbrot fractal. We hoped to construct something resembling both a twill weaving fashion pattern and a Mandelbrot set, but that did not happen.

We found a new fractal, which we explored and developed further to design and implement an attractive fashion item. A preview of the new fractal's different aesthetic qualities is presented in Figure 1.

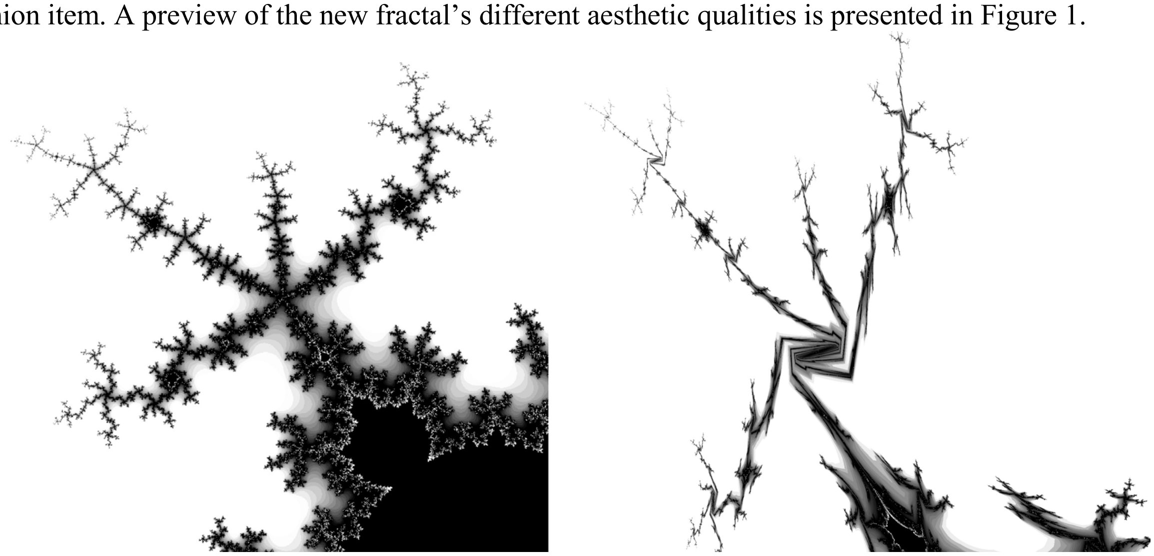


Figure 1: *Distinctive aesthetic qualities of comparable antenna objects in the classic Mandelbrot set (left) and the novel modified Mandelbrot-like set (right).*

The plan of the work is as follows: first, we summarize the equations of the classic Mandelbrot set, using $\mathbb{C}$, the set of complex numbers. Complex numbers can be added and multiplied. But we shall modify the way multiplication works, adopting another metric: the Taxicab metric. Then we describe our coding which we use to implement the new multiplication and thus make a new Mandelbrot-like fractal. Then we explore the visual features of the result. Finally, we apply the results in a fashion setting.

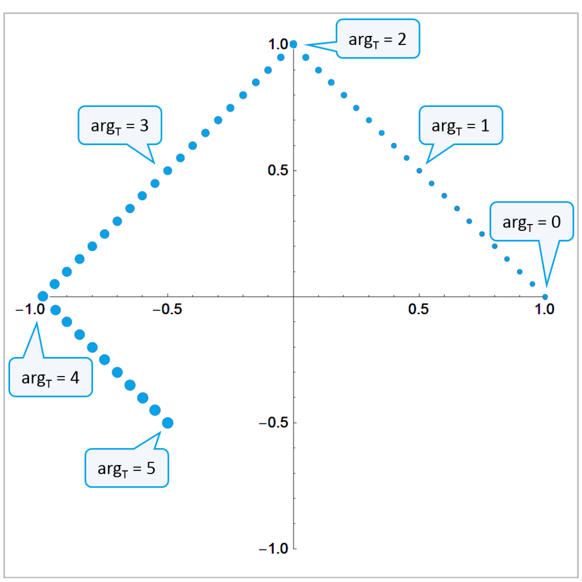
An Unconventional Multiplication

We assume that the reader is familiar with complex numbers. We summarize the main facts and, for later usage, fix our notation. An arbitrary complex number can be written as $x + iy$ where both x and y are real numbers. The special number i has the property $i^2 = -1$, in other words, $i = \sqrt{-1}$. The ordinary rules of algebra apply, e.g. $(x_1 + iy_1) \times (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

These algebraic definitions have a geometric interpretation if, following Argand, we interpret the numbers x and y as Cartesian coordinates in a two-dimensional plane. Each complex number is a point, or equivalently, a vector starting from the origin. Complex number addition is simply vector addition. For the geometric interpretation of multiplication, it is helpful to use polar coordinates, so each complex number z has an angle ϑ and a distance r to the origin (O). We write $\arg(z) = \vartheta$ and $\text{abs}(z) = r$. If $\arg(z_1) = \vartheta_1$ and $\text{abs}(z_1) = r_1$ and similarly $\arg(z_2) = \vartheta_2$ and $\text{abs}(z_2) = r_2$, then $\arg(z_1 \times z_2) = \vartheta_1 + \vartheta_2$ and $\text{abs}(z_1 \times z_2) = r_1r_2$. So we add the angles and multiply the lengths. Angles are in radians: the angle of a complex number z is obtained by first normalizing the number: finding the intersection of the line Oz and the unit circle. Then $\arg(z)$ is the distance from $1 + 0i$ to the normalized number, traveling counter-clockwise along the unit circle. In this way, multiplication is defined geometrically, only using distance. This geometric definition is important for our next step.

Recall the traditional definition of the Mandelbrot fractal $\mathcal{M}$ [5]. The set $\mathcal{M}$ contains precisely those points c for which repeated iteration of the quadratic mapping $z \mapsto z^2 + c$ is bounded (the first value of z is defined as $z_0 = 0$). In the visual appearance of the Mandelbrot set, the circles and its relatives occur and re-appear in all approximations and at all zoom-in levels. With "relatives of the circle" we refer to cardioids, spirals, lemniscates, and so on. This is no surprise since in the heart of the Mandelbrot fractal's definition there is a mapping $\mapsto z^2 + c$, one addition and one multiplication, the latter operation being geometrically based on the unit circle.

This multiplication can be replaced by an alternative if we change the underlying notion of distance, that is, the metric [6]. A metric on $\mathbb{C}$ is a mapping $\mathrm{d}: \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ satisfying three requirements: $\mathrm{d}(z_1, z_2) = 0$ if and only if $z_1 = z_2$ (the identity axiom), $\mathrm{d}(z_1, z_2) + \mathrm{d}(z_2, z_3) \geq \mathrm{d}(z_1, z_3)$ (the triangle axiom), and $\mathrm{d}(z_1, z_2) = \mathrm{d}(z_2, z_1)$ (the symmetry axiom). Here we adopt the Taxicab metric, also known as Manhattan metric or 1st order Minkowski distance. If $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$ then $\mathrm{d_T}(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2|$, replacing the Euclidean $\mathrm{d_E}(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

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Algorithm for given cx, cy:

x = 0
y = 0
s = 0
α = 0
i = 0
while i < BAILOUT and s ≤ 2
    s = s × s
    α = argT(x,y)
    α = α + α
    x = cx + s × cosT(α)
    y = cy + s × sinT(α)
    s = |x| + |y|
    i++
return i
```

Figure 2: *Angle based on the unit square (left) and algorithm for the Taxicab Mandelbrot (right).*

Now what is the set of all points equidistant to the origin? It is no longer a circle, but a square rotated by 45 degrees with respect to the Cartesian grid. It is the set of points $z = x + iy$ such that $|x| + |y| = 1$. The concept of a unit circle has been replaced by a unit square. Whenever we take two numbers z_1 and z_2 on the unit square, their product $z_1 \times_{\mathrm{T}} z_2$ is the unique point on the unit square such that $\arg_{\mathrm{T}}(z_1 \times_{\mathrm{T}} z_2) = \arg_{\mathrm{T}}(z_1) + \arg_{\mathrm{T}}(z_2)$. Here we take the "angle" $\arg_{\mathrm{T}}(z)$ to be the distance from $1 + 0i$ to the normalized number, *traveling along the unit square*. If a point z is not on the unit square we can find its angle by first normalizing the number, that is, by finding the intersection of the line Oz and the unit square. The distance from $z = x + iy$ to the origin is denoted as $\mathrm{abs}_{\mathrm{T}}(z)$ but we use the Taxicab metric, so $\mathrm{abs}_{\mathrm{T}}(z) = |x| + |y|$. Figure 2 (left) presents an example: for $z = x + iy = -½ - ½i$ we find $\mathrm{abs}_{\mathrm{T}}(z) = 1$ and $\arg_{\mathrm{T}}(z) = 5$.

A motivation for the Taxicab distance function is that the idea of anisotropy (all properties being direction-independent) holds in certain fields of human experience and science, but not in others. In our earlier projects, we took inspiration from weaving, a discipline in which neither the construction nor the emergent fabric properties satisfy anisotropy. In weaving there are two distinct directions: the warp and the weft; all yarns run either horizontally or vertically – nothing else. To measure the distance between points z_1 and z_2, one must travel some distance horizontally, then vertically and add up those distances.

Implementation

The above ideas are implemented by coding the definitions of + and $\times_{\mathrm{T}}$. Working in Mathematica, we need in fact only code + and the squaring operator $\mathrm{sq}_{\mathrm{T}}(z) = z \times_{\mathrm{T}} z$, since these are all that is necessary for re-implementing the Mandelbrot set. The auxiliary operations $\cos_{\mathrm{T}}$, $\sin_{\mathrm{T}}$ are defined for positive "angles" φ:

$$\cos_{\mathrm{T}}(\varphi) = 1 - \tfrac{\varphi}{2} \quad (\text{if } \varphi < 4)$$
$$\cos_{\mathrm{T}}(\varphi) = \tfrac{\varphi}{2} - 3 \quad (\text{if } 4 \leq \varphi < 8)$$
$$\cos_{\mathrm{T}}(\varphi) = \cos_{\mathrm{T}}(\varphi - 8) \ (\text{otherwise})$$

$$\sin_{\mathrm{T}}(\varphi) = \tfrac{\varphi}{2} \quad (\text{if } \varphi < 2)$$
$$\sin_{\mathrm{T}}(\varphi) = 2 - \tfrac{\varphi}{2} \quad (\text{if } 2 \leq \varphi < 6)$$
$$\sin_{\mathrm{T}}(\varphi) = \tfrac{\varphi}{2} - 4 \quad (\text{if } 6 \leq \varphi < 8)$$
$$\sin_{\mathrm{T}}(\varphi) = \sin_{\mathrm{T}}(\varphi - 8) \ (\text{otherwise})$$

Although Mathematica knows how to work with complex numbers, we let our definitions work on pairs of real numbers, which gives us a bit more control, for example on precision. So $\arg_{\mathrm{T}}(x, y)$ implements $\arg_{\mathrm{T}}(x + iy)$. We set $\arg_{\mathrm{T}}(0,0) = 0$. When $(x, y) \neq (0,0)$ we set:

$$\arg_{\mathrm{T}}(x, y) = 1 - \frac{x-y}{x+y} \quad (\text{if } x \geq 0 \text{ and } y \geq 0)$$
$$\arg_{\mathrm{T}}(x, y) = 3 - \frac{x+y}{-x+y} \quad (\text{if } x \leq 0 \text{ and } y \geq 0)$$
$$\arg_{\mathrm{T}}(x, y) = 5 + \frac{x-y}{-x-y} \quad (\text{if } x \leq 0 \text{ and } y \leq 0)$$
$$\arg_{\mathrm{T}}(x, y) = 7 + \frac{x+y}{x-y} \quad (\text{otherwise})$$

The number BAILOUT in Figure 2 (right) is usually set to 100, but it can be adjusted to influence the visual appearance of the fractal's approximation. It is common practice to generate approximations of $\mathcal{M}$ by calculating a certain number of iterations z_0, $z_1, z_2 \ldots z_{\mathrm{BAILOUT}}$ while checking whether $|z| \leq 2$. We do precisely the same, except for using $z \mapsto \mathrm{sq}_{\mathrm{T}}(z) + c$ and checking whether $\mathrm{abs}_{\mathrm{T}}(z) \leq 2$.The escape criterion $|z| \leq 2$ is appropriate for the taxicab set: the proof that the threshold is safe, given on [10] page 738, only uses $|z^2| = |z|^2$ and the triangle inequality, both of which still hold.

The "quadratic" mapping $f_c: \mathbb{C} \to \mathbb{C}$ is defined by $f_c(z) \mapsto \mathrm{sq_T}(z) + c$ and the iteration begins at $z = 0$. Instead of the fractal $\mathcal{M}_\mathrm{T}$ itself, we calculate approximations where we consider only n-fold iteration for n-values up to the limit BAILOUT. The boundaries of the sets thus obtained, for successive iteration counts, are traditionally referred to as *lemniscates*. For the classic Mandelbrot set these are first a circle, then a Cassinian oval, then a pear curve, after which they get more and more complex, yet smooth. For the new fractal, the first "lemniscate" is the unit square. The second "lemniscate" consists of 14 pieces. Each "lemniscate" has a piecewise polynomial equation (we found the explicit formula for the second, Figure 3).

$$\begin{cases} x + x^2 + y + 2xy + y^2 == 2 & (x \ge 0 \,\&\&\, x \ge y) \\ -x + x^2 + y + 2xy + y^2 == 2 & (x \ge 0 \,\&\&\, x < y \,\&\&\, x + x^2 - y^2 < 0) \\ x + 3x^2 + y + 2xy - y^2 == 2 & (x \ge 0 \,\&\&\, x < y \,\&\&\, x + x^2 - y^2 \ge 0) \\ -2x^2 - x\,(1 + x) + y + 2xy + y^2 == 2 & (x < 0 \,\&\&\, x + y \ge 0 \,\&\&\, -2x^2 + y + 2xy > 0) \\ 2x^2 - x\,(1 + x) - y - 2xy + y^2 == 2 & (x < 0 \,\&\&\, x + y \ge 0 \,\&\&\, -2x^2 + y + 2xy \le 0) \\ -x - x^2 - (1 + 2x - 2y)\,y + y^2 == 2 & (x < 0 \,\&\&\, x + y < 0 \,\&\&\, x + x^2 - y^2 < 0) \\ x + x^2 - 2xy + (-1 + y)\,y == 2 & (x < 0 \,\&\&\, x + y < 0 \,\&\&\, x + x^2 - y^2 \ge 0) \end{cases}$$

Figure 3: *Piecewise polynomial equation for the second "lemniscate", 7 segments for $y > 0$.*

The new fractal is shown in Figure 4, where it is rendered using Mathematica's ArrayPlot, showing 20 iteration counts. We sweep twice through the hue range: 0=red, 1=orange, etc. till 9=red (and again). The rest, which includes $\mathcal{M}_\mathrm{T}$, is rendered black or grey.

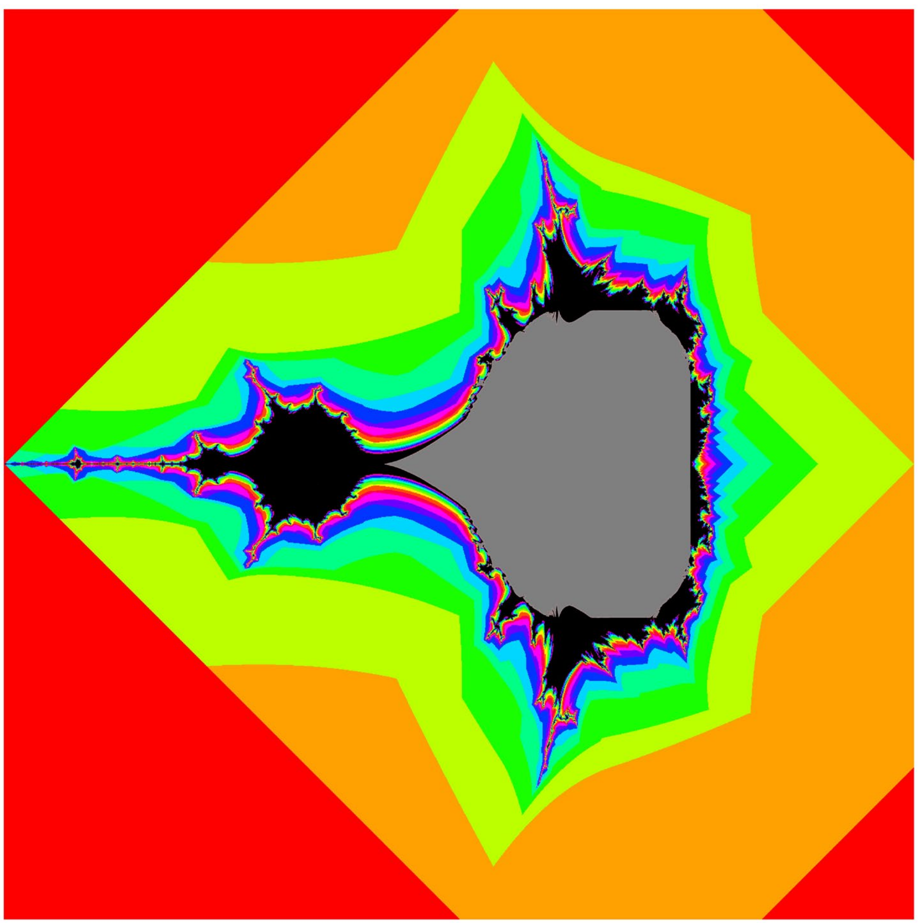

Figure 4: *Taxicab-inspired Mandelbrot-like fractal.*

The new fractal appears as a morphed version of the classic. It is not a simple global morphing, as we find similar, yet different structures at all zoom-in levels.

First, we summarize some features of the classic fractal, all of which are well-known. Then we can state which features are or are not preserved in the new fractal. The classic fractal has a main cardioid and around it an infinite number of so-called bulbs. The bulbs are numbered by a coding schema: each bulb is given a fractional form m/n. For example, the biggest bulb, which is circular, left of the main cardioid, is the 1/2 bulb. The relative positions of the bulbs are described by Farey addition, see Devaney [5]. The antenna of an m/n bulb branches into n sub-branches.

These are the main features shared between our new fractal and the classic fractal: the new fractal has bulbs with the same coding scheme and Farey addition still works. There are period-doubling bulbs for each bulb. There are antennas and their branching corresponds again to the denominator of the bulb's fractional form. There are also mini versions of the main fractals inside the antennas.

These are the main differences: the new main set is no longer a cardioid. The bulbs are no longer connected to the main figure via a single point: some bulbs are partially implanted like a tooth (for example the 1/3, 1/4, 1/5 etc. bulbs) whereas other bulbs appear connected via a stick, like a poppy capsule on a stem (for example the 2/5 bulb). Whereas the classic fractal has a non-smooth boundary everywhere, this seems not to be the case for the new fractal. We conjecture that the neck connecting the main "cardioid" and the 1/2 bulb, near $\mathrm{Re}(z) \approx -0.6$, is smooth. The appearance of features is more angular, spiky, aggressive or spooky (preliminary experiments suggest the same for the filled Julia sets). We use "Iterates for the Mandelbrot Set" (demonstrations.wolfram.com) by Felipe Dimer de Oliveira, as a tool to study orbits and thus confirm the presence of period-doubling secondary bulbs and test the depth of the implanted 1/3 bulb. The points whose orbit has a single fixed point, the "cardioid", form the inner (grey) core in Figure 4.

Zooming in to details, we calculate the number of iterations again, usually a number between 1 and 100. In Figure 4 this is color-coded in 100 colors, using one sweep for the hue range (*H)*. The numbers beyond 99 represent the best approximation of the fractal itself and are rendered black.

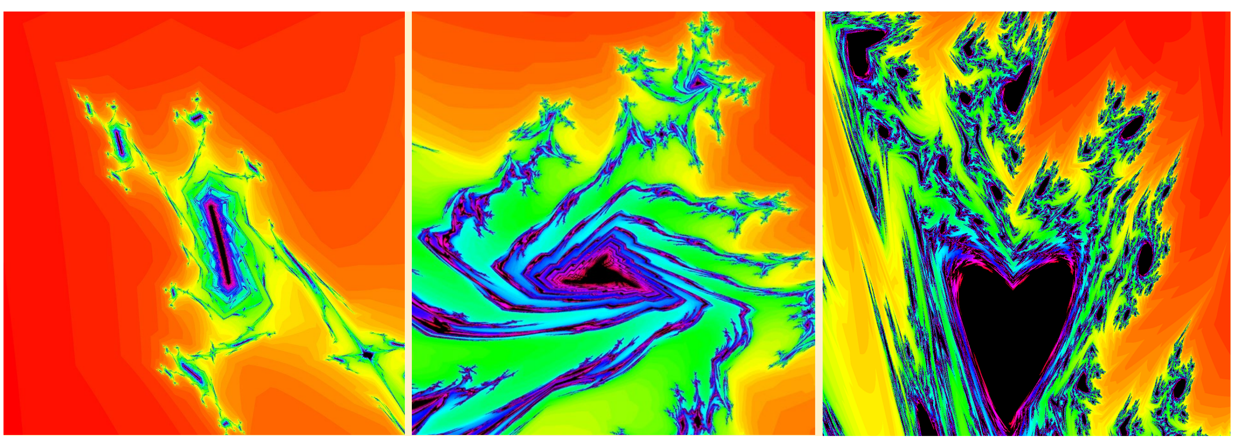

Figure 5: *Button hole island, triangles archipelago, and heart archipelago.*

It is a rewarding experience to navigate in this unknown territory, looking for visually exciting features. A 3000×3000 pixel image takes about 10 minutes to generate on a 12 core server with 32MB RAM. We highly recommend this type of research because it makes one feel being on an adventure, like being captain James Cook. Any classical fact may or may not hold and we have to re-read the theory on the Mandelbrot fractal, re-calculate every feature, re-consider every conjecture. Figure 5 shows a few objects we found.

Towards Fashion

In fashion, one can choose to work at the level of fabric motifs, accessories, garments, or even complete collections. Here we aim at a garment. Although the button-hole island is like an invitation to develop new closures and buttoning solutions, we leave that as an option for future research. The present goal is to have a meaningful mapping of $\mathcal{M}_{\mathrm{T}}$ features to garment features. Therefore we explored several garment construction principles, as shown in Figure 6. Where to cut-out the fractal and where to leave it as white, is a design parameter during pattern-cutting, and so is the interplay of seam-lines, cut-lines, and lemniscates.

Figure 6: *Garment construction principles: sewing together along lemniscates gives protruding volumes (left), removing the main "cardioid" leaves an elegant neck opening with lace-like qualities (center), cutting along a lemniscate could give a natural seam for joining a sleeve (right).*

Which colors should we use? In fashion, the idea of skin color, sometimes called "nude" is a theme of growing interest. There are multiple approaches toward skin color, for example by Pantone and Rihanna, mostly aimed at make-up, or by medical researchers, for example, Azmi [2] aiming at simulation babies. We took inspiration from fashion designer Christian Louboutin. In 2015 he released a collection of nude shoes using five colors to represent the diversity of the women buying his products. In 2017 he added two new colors.

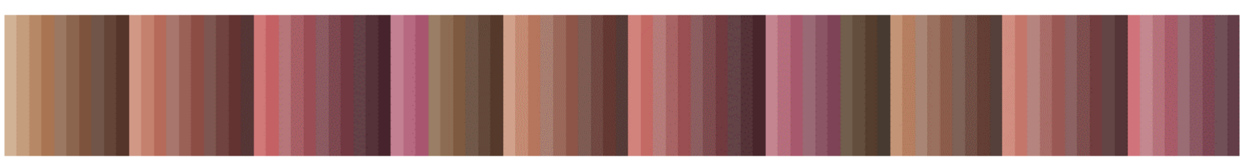

Figure 7: *Traversing the nude color space in 99 steps.*

We did not copy the shoe colors but studied the color space of the models in a photo of his campaign, see Jones [8]. We found an online 3519×2374px image by Sofia Mauro in Jones [8]. We sampled the legs of the seven girls systematically on four spots (shiny spot, shadow spot, normal spot upper leg, normal spot lower leg), which gave us 28 HSB values, using Photoshop's color picker. In HSB (Hue Saturation Brightness) color space, we found that H and B have a positive correlation. H and S correlate negatively and so do B and S. But instead of taking a linear color range, we constructed a sequence of 99 HSB values moving zigzag-wise through the 3D color space spanned by the range of H, S, and B ranges. This gave us a color palette which is even richer than the human skin tones. We have 3 sweeps in the H dimension, 33 for S and 10 sweeps for B, see Figure 7.

Adopting a BAILOUT value of 100, we mapped the iteration count to the first 99 colors, except for iteration count 100, which is mapped to white (saving ink during printing, sometimes we cut-out the "cardioid"). One of the first experimental garments is shown in Figure 8.

We cut out the area $-2 < \mathrm{Re}(z) < 1, -1.5 < \mathrm{Im}(z) < 1.5$ and scaled this to 100×100cm and to 70×70cm. A coarse grid and coordinate numbers are added, a subtle techno-detail which, for an earlier version, was praised and appreciated by several of our designer-colleagues and students. The scaled images were printed on Jetcol® dye sublimation paper and transferred to large pieces of white woven fabric. The large-scaled fabric is meant for the body, the small-scaled fabric for the sleeves.

Related Work and Concluding Remarks

The Mandelbrot set has been an icon for the burgeoning field of chaos theory (Horgan [7]). Arthur C. Clarke has called it one of the most beautiful and remarkable discoveries in the entire history of mathematics.

Already in 1994 Jhane Barnes has used sections of Mandelbrot sets and Julia sets, arranged into wallpapers, to create Jacquard fabrics for garments, see Amidon [1]. Bayar et al. [3] studied the relationship between complex numbers and the Taxicab plane, but although they changed the distance metric, they would still measure angles in radians, as numbers between 0 and 2π. Here we decided not to do that, but measure angles along the new unit square (although we probably destroy most of the old algebraic properties of the number system, the algorithm still defines a set).

Recently, Blankers et al. [4] defined a modified Mandelbrot set, replacing the complex numbers $\mathbb{C}$ by hyperbolic numbers $\mathbb{H}$. The resulting parameter set $\mathcal{M}_{\mathbb{H}}$ does not appear fractal, some of its Julia sets do.

Figure 8: *Experimental top and trousers based on new Mandelbrot-like fractal in nude colors. Photo Daisy van Loenhout, model Cindy Tieleman, ©Marina Toeters.*

We found a way to generalize the classic fractal's definition and thus found ourselves thrown into the adventure of discovering the properties of a new fractal. We established a few facts and conjectures about the new set $\mathcal{M}_{\mathrm{T}}$, but there are still open questions (for example, is the fractal connected?) The fractal presented here is only one out of a large collection of new possibilities, as the boundary of any point-symmetric concave set is a possible unit "circle" (Minkowski's idea). Besides $\mathrm{d_T}$ we tested $\mathrm{d_M}(z_1, z_2) = \max(|x_1 - x_2|, |y_1 - y_2|)$, which gives another fractal, different from $\mathcal{M}_{\mathrm{T}}$, yet featuring similar angular aesthetics.

We also found a new direction of exploiting this type of fractal in fashion, avoiding the fractal to be just an add-on nor resorting to wallpaper-like repetition.

Acknowledgments

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References

[1] C. Amidon (2002). "Materializing the Invisible." *Surface Design Journal* (Summer 2002) pp. 28−33. http://www.jhanebarnes.com/assets/surfacedesignjournal_summer2002_sm.pdf

[2] N. F.B. Azmi (2018). "Color correction of baby images for cyanosis detection." *Proc. 22nd MIUA Conference.* Springer Communications in Computer and Information Science, Vol. 894, pp. 354–370.

[3] A. Bayar, S. Ekmekçi, İ. Öztürk (2015). "On Complex Numbers and Taxicab Plane." *Mathematical Sciences and Applications E-Notes*, *3*(1), pp. 58−64.

[4] V. Blankers, T. Rendfrey, A. Shukert, P.D. Shipman (2019). "Julia and Mandelbrot Sets for Dynamics over the Hyperbolic Numbers." *arXiv*:1901.02085v1 [math.DS].

[5] R.L. Devaney (1999). "The Mandelbrot set, the Farey tree, and the Fibonacci sequence." *The American Mathematical Monthly* 106, no. 4, pp. 289−302.

[6] M. Hazewinkel Ed. (2001). "Metric space." *Encyclopedia of Mathematics*, Springer Netherlands.

[7] J. Horgan (1990). "Mandelbrot set-to." *Scientific American* 262, no. 4, pp. 30−35.

[8] C. Jones (2017). "Celebs are clamoring for these nude heels." *New York Post*. https://nypost.com/2017/06/28/celebs-are-clamoring-for-these-nude-heels/

[9] B. Mandelbrot (1982). *The fractal geometry of nature* (Vol. 1). New York: WH freeman.

[10] H. Peitgen, H. Jürgens, and D.Saupe, *Chaos and Fractals: New Frontiers of Science.* Springer Verlag 1992.